Graph Theory

Exercises Bank

McGill University

By Jiajun Zhang

October 13, 2025

Contents

Graph Theory Selected Exercises

1.1 Graph Basics: Connectivity

Exercise

Exercise 1. *The compliment of a graph G, denote by G^c, is defined by the following: If $u, v \in V(G)$ are adjacent in G, then they are not adjacent in G^c; If $u, v \in V(G)$ are not adjacent in G, then they are adjacent in G^c. Show either G or G^c is connected.*

Proof. If $E(G) = \varnothing$ or G is connected, then we win. It suffices to show that G is not connected but $E(G) \neq \varnothing$ will imply G^c is connected. Take any $u, v \in V(G)$, then:
(i) If u, v are not adjacent, then by definition u, v is adjacent in G^c and hence u, v are connected;
(ii) If u, v are adjacent, then they belong to the same connected component C. Since we assume G is not connected, so $\exists w \in V(G \setminus C)$, hence neither u nor v is adjacent to w, and hence in G^c, u, w are adjacent, v, w are adjacent. So there is a path $u \to w \to v$ and hence u, v are connected. Hence G^c is connected. ■

Exercise

Exercise 2. *Let T be a tree with n vertices, if there exists $v \in V(T)$ with $\deg_T(v) = k$, show T has at least k leaves.*

Proof. If $n = 0$ then there is nothing to show. We assume T has at most $k - 1$ leaves, and we can even let k be the largest degree. Then by handshaking lemma we have

$$\begin{aligned} 2|E(T)| &= \sum_{v \in V(T)} \deg(v) && (1) \\ &\leq k - 1 + \sum_{v \in V(T), \deg(v) > 1} \deg(v) && (2) \\ &\leq k - 1 + (n - k + 1)k && (3) \end{aligned}$$

also we know T is a tree so $|E(T)| + 1 = |V(T)|$ and we indeed have $2n - 2 \leq k - 1 + (n - k + 1)k$. Fix n and we have

$$2n - 2 \leq -k^2 + (n + 2)k - 1$$

as a function of k where $k \in \mathbb{N}$. Let $f(k) = -k^2 + (n + 2)k - 1$, it is a parabola which obtains its maximum at $k = n + 2$. However $f(2n + 2) = -(n + 2)^2 + (n + 2)(n + 2) - 1 = -1$ and $2n - 2 \leq -1$ iff $n = 0$, a contradiction. ■

Exercise

Exercise 3. *Let G be a simple graph such that each vertex has degree at least k. Show that G has a cycle with length at least k.*

Proof. Let P denote the longest path in G and v to be one end. We claim that all neighbors of v must be on P otherwise we can make the path longer. Then denote u to be the furthest neighbor of u on the path P, then along the path P the distance from u to v is $k-1$, plus u is connected to v by an edge e, then P along with e is the path of desired length (shortest). ∎

Exercise

Exercise 4. *Denote $g(G)$ to be the length of the shortest cycle in G. Show that if G contains a cycle then $g(G) \leq 2diam(G) + 1$.*

Proof. Assume such a shortest cycle has length at least $2\text{diam}(G) + 2$, then $\exists u, v \in V(C)$ such that their distance on C is at least $\text{diam}(G) + 1$. Then use the definition in G any path P with ends u, v has length at most $\text{diam}(G)$ and hence it follows that $P \subsetneq C$ and hence P along with the shorter $x - y$ path in C will form a shorter circle and that's a contradiction (overlapping of vertices are possible but that will make our cycle even shorter). ∎

1.2 Spanning Trees and Minimum Spanning Trees

Exercise

Exercise 5. *Let G be a connected graph such that*

1.3 Bipartite Graphs and Matching

Exercise

Exercise 6. *Let G be a connected bipartite graph, and $3 \leq \deg u \leq k$ for all vertex $u \in V(G)$. Prove that G has a matching of size at least $\frac{3|V(G)|}{2k}$.*

Proof. Let (A, B) be a fixed partition of $V(G)$, then by definition

$$|E(G)| = \sum_{u \in A} \deg u = \sum_{v \in B} \deg v$$

and we more over have

$$3|A| \leq |E(G)| \leq |A|k \quad \text{and} \quad 3|B| \leq |E(G)| \leq |B|k$$

so we have

$$3|A| + 3|B| \leq 2|E(G)| \leq k|A| + k|B|$$

which is

$$\frac{3|V(G)|}{2} \leq |E(G)| \leq \frac{k|V(G)|}{2}.$$

Let X be any vertex cover, then $|E(G)| \leq k|X|$, hence

$$k|X| \geq \frac{3|V(G)|}{2} \implies |X| \geq \frac{3|V(G)|}{2k}$$

Using König's theorem, $\nu(G) = \tau(G)$, hence choose X to be minimum, we have $\nu(G) \geq \frac{3|V(G)|}{2k}$ and hence it has a matching of at least that size.

$\blacksquare$

Exercise

Exercise 7. *If G is connected with no cut-edge, and each vertex has degree exactly 3, prove that any two edges in $E(G)$ lie on a common cycle.*

Proof. Take any two edges, wlog, let e_1 has ends u_1, v_1 and e_2 has ends u_2, v_2. Let $A \subseteq G$ contains only e_1 and its two ends; $B \subseteq G$ contains only e_2 and its two ends. Then choose $k = 2$, by Menger's second theorem, we have two cases:

(i) There exist 2 vertex distinct paths P_1, P_2 from A to B. In this case by relabeling $P_1 \cup \{e_1\} P_2 \cup \{e_2\}$ will be the desired cycle.

(ii) There exist a separation of order 1. That is, we let $u \in V(G)$ be such a vertex, let $A \subseteq P, B \subseteq Q$, then since $\deg u = 3$ wlog it has one neighbor in P and two neighbors in Q. Then the edge with one end in u and the other end in P is a cut-edge, which is a contradiction so this cannot happen. $\blacksquare$