Multi-variable Calculus Sample Examination

• INSTRUCTIONS •

(i) This exam paper has two parts, part (i) will have 12 multiple choice questions, each question will carry 5 marks; part (ii) will have 5 long answer questions, each question will carry 20 marks. You will be given 3 hours to finish the exam, and the total mark for this exam is 160.

(ii) In order to match the purpose of this course, the difficulty of the questions are randomly distributed, meaning the difficulty in each part **is not exactly** from easy to hard. Don't panic if you encounter hard questions, try your best and the grading will be generous overall.

PART (i)

Multiple Choice Questions : Each question will have exactly one correct answer.

1. The value of $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$ is

(A) $\frac{1}{2}$ (B) 2 (C) 0 (D) The limit does not exist

2. Let $\boldsymbol{r}(t)$ be a vector function representing the intersection of $x^2+y^2=1$ and $y+z=2$ in $\mathbb{R}^3$. Fix $t\in[0,2\pi]$, what is the representation of $\boldsymbol{r}(t)$?

(A) $\boldsymbol{r}(t)=\langle\cos(t),\sin(t),2-\sin(t)\rangle$ (B) $\boldsymbol{r}(t)=\langle\sin(t),\cos(t),2-\sin(t)\rangle$

(C) $\boldsymbol{r}(t)=\langle\cos(t),\sin(t),2-\cos(t)\rangle$ (D) $\boldsymbol{r}(t)=\langle\cos(t),\sin(t),2-\sin(t)\rangle$

3. The vector function is given by $\boldsymbol{r}(t)=\langle 1+t^3, te^{-t}, \sin(2t)\rangle$. What is the unit tangent vector of $\boldsymbol{r}(t)$ at $t=0$?

(A) $\langle 0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$ (B) $\langle 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle$ (C) $\langle 0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle$ (D) $\langle 0, \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\rangle$

4. The torsion of a parameterized curve $\boldsymbol{r}(t)$ is given by

$$\boldsymbol{\tau}(t) = -\frac{\mathcal{B}'(t) \cdot \mathcal{N}'(t)}{\|\boldsymbol{r}'(t)\|}$$

where $\mathcal{N}(t) = \dfrac{\mathcal{T}'(t)}{\|\mathcal{T}'(t)\|}$, $\mathcal{T}'(t)$ is the normal vector of $\boldsymbol{r}'(t)$ and $\mathcal{B}(t) = \mathcal{T}(t) \times \mathcal{N}(t)$. Then what is the torsion of $\boldsymbol{r}(t) = \langle \cos(t), \sin(t), t \rangle$?

(A) 2 (B) -2 (C)$-\dfrac{1}{2}$ (D)$\dfrac{1}{2}$

5. The tangent plane to $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9} = 3$ at the point $(-2, 1, -3)$ is given by

(A) $6x - 3y + 2z + 18 = 0$ (B) $3x - 6y + 2z - 18 = 0$

(C) $6x - 2y - 3z + 18 = 0$ (D) $2x - 6y + 3z + 18 = 0$

6. Suppose $z = e^x \sin(y), x = st^2, y = s^2 t$, then $\dfrac{\partial z}{\partial s} + \dfrac{\partial z}{\partial t}$ is

(A) $e^x(\sin(y)t^2 + \cos(y)s^2) + 2e^x st(\sin(y) + \cos(y))$

(B) $e^x(\sin(y)s^2 + \cos(y)t^2) - 2e^x st(\sin(y) + \cos(y))$

(C) $e^x(\sin(y)s^2 + \cos(y)t^2) + 2e^x st(\sin(y) + \cos(y))$

(D) $e^x(\sin(y)t^2 + \cos(y)s^2) - 2e^x st(\sin(y) + \cos(y))$

7. In $\mathbb{R}^2$, a curve is given by $\mathcal{C} : \dfrac{x^2}{2} + y^2 + 2xy - 1 = 0$. Suppose $\mathbf{M}$ is a point on $\mathcal{C}$, then what are the closest and furthest distance from $\mathbf{M}$ to the origin $(0, 0)$?

(A) Closest : $\dfrac{2}{7}$; Furthest : 2 (B) Closest : $\sqrt{\dfrac{2}{7}}$; Furthest : $\sqrt{2}$

(C) Closest : $\sqrt{\dfrac{4}{7}}$; Furthest : $\sqrt{2}$ (D) Closest : $\dfrac{4}{7}$; Furthest : 2

8. The double integral $\displaystyle\iint_{\mathcal{D}} (x^2 + y^2) dxdy$ bounded by $\mathcal{D} = \{1 \leq x^2 - y^2 \leq 9, 2 \leq xy \leq 4\}$ is

(A) 16 (B) 8 (C) 16 $\sqrt{2}$ (D) $8\sqrt{2}$

9. Suppose a 2-D disk on the xy plane is the shape given by $1 \leq x^2 + y^2 \leq 4, x \geq 0, y \geq 0$. The density at each point is proportional to its distance to the origin. Denote $\boldsymbol{\rho}(x, y)$ to be the density function at each point, then the center of mass of the disk $\mathbf{M}(\overline{x}, \overline{y})$ is given by

$$\overline{x} = \frac{\iint_{\mathcal{D}} x\boldsymbol{\rho}(x,y)d\mathbf{A}}{\iint_{\mathcal{D}} \boldsymbol{\rho}(x,y)d\mathbf{A}} \quad \text{and} \quad \overline{y} = \frac{\iint_{\mathcal{D}} y\boldsymbol{\rho}(x,y)d\mathbf{A}}{\iint_{\mathcal{D}} \boldsymbol{\rho}(x,y)d\mathbf{A}}$$

What is the center of mass of this disk?

(A) $(0, \frac{28}{9\pi})$ (B) $(0, \frac{5}{\pi})$ (C)$(0, 0)$ (D) $(0, \frac{45}{14\pi})$

10. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be a vector field, $\boldsymbol{c}(t) = \langle t, t^2, t^2 \rangle$ where $0 \leq t \leq 1$. Then what is the value of the line integral along the curve $\boldsymbol{c}(t)$ given by $\int_c \mathbf{F}d\mathbf{s}$?

(A) $\frac{3}{2}$ (B) $-\frac{3}{2}$ (C) $\frac{7}{5}$ (D) $-\frac{7}{5}$

11. Compute the surface integral $\iint_{\mathbf{S}} (x + z^2)d\mathbf{S}$ where $\mathbf{S} := \{x^2 + y^2 - z^2 = 0, z \in [1, 4]\}$.

(A) $\frac{255}{2}\pi$ (B) $\frac{255}{4}\pi$ (C) $\frac{255\sqrt{2}}{2}\pi$ (D) $\frac{255\sqrt{2}}{4}\pi$

12. Let $\mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} \in \mathbb{R}^2 \backslash \{(0, 0)\}$ be a vector field. Two parameterized curves are given by $\boldsymbol{c}_1(t) = \langle 3 + 2\cos(t), -2 + \sin(t) \rangle$, and $\boldsymbol{c}_2(t) = \langle 2\cos(t), \sin(t) \rangle$ for all $t \in [0, 2\pi]$. Then the line integral along the curves $\boldsymbol{c}_1(t), \boldsymbol{c}_2(t)$ are

(A) $\oint_{\boldsymbol{c}_1} \mathbf{F}d\mathbf{s} = 0$ and $\oint_{\boldsymbol{c}_2} \mathbf{F}d\mathbf{s} = 2\pi$ (B) $\oint_{\boldsymbol{c}_1} \mathbf{F}d\mathbf{s} = 0$ and $\oint_{\boldsymbol{c}_2} \mathbf{F}d\mathbf{s} = 0$

(C) $\oint_{\boldsymbol{c}_1} \mathbf{F}d\mathbf{s} = \pi$ and $\oint_{\boldsymbol{c}_2} \mathbf{F}d\mathbf{s} = \pi$ (D) $\oint_{\boldsymbol{c}_1} \mathbf{F}d\mathbf{s} = 2\pi$ and $\oint_{\boldsymbol{c}_2} \mathbf{F}d\mathbf{s} = 0$

PART (ii)

Long Answer Questions : Each question requires detailed calculations, proofs, or reasoning

13. (Differentiation)

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$f(x,y) = \frac{(x-1)y}{(x-1)^2 + y^2}$$

Show that $f(x,y)$ is continuous at $(1,0)$.

14. (Extreme Values of Multi-Variable Functions)

Find the maximum value of $f(x,y) = \sin(x)\sin(y)\sin(x+y)$ over the closed subset of $\mathbb{R}^2$ given by

$$\mathcal{D} := \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x + y \leq \pi\}$$

And point out at which point(s) of $\mathcal{D}$ is the maximum attained.

15. (Parameterized Curves)

Determine a function $g(t)$ such that the curve $\boldsymbol{c}(t) = \langle t, g(t) \rangle$ passes through the point $(1,1)$ and intersects the level curve of $f(x,y) = x^4 + y^2$ everywhere at the same angle.

16. (Triple Integral)

Compute the triple integral $\iiint_{\mathcal{A}} \left(\frac{x^2}{2} + \frac{y^2}{3} + \frac{z^2}{5}\right) dxdydz$, where $\mathcal{A}$ is bounded by

$$\mathcal{A} := \{(x,y,z) \in \mathbb{R}^3 | \frac{x^2}{2} + \frac{y^2}{3} + \frac{z^2}{5} \leq 1\}.$$

17. (Vector Calculus)

Given that $\mathbf{F}$ is a vector field given by $\mathbf{F} = (x^2 + y - 4)\mathbf{i} + (3xy)\mathbf{j} + (2xz + z^2)\mathbf{k}$. A region $\mathcal{S} := \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 16, z \leq 0\}$, with the direction of the norm pointing outwards. Find $\iint_{\mathcal{S}} (\boldsymbol{\nabla} \times \mathbf{F}) d\mathbf{S}$.