Honors Complex Analysis

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1 Introduction to complex numbers

1.1 Basic definitions

Definition 1. *The set of complex numbers, $\mathbb{C}$, is defined by*

$$\mathbb{C} := \{x + yi | x, y \in \mathbb{R}\},$$

where $i^2 = -1$.

Like real numbers, $\mathbb{C}$ is equipped with basic operations. Let $z = a + bi, w = c + di$, then

- $z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$,
- $zw = (a + bi)(c + di) = (ac - bd) + (bc + ad)i$.
- We have $|z \cdot w| = |z| \cdot |w|$. Verify it yourself.
- Suppose $(a, b) \neq (0, 0)$, then the *inverse* of z is defined by

$$z^{-1} = \frac{1}{z} = \frac{1}{a + bi} = \frac{(a - bi)}{(a + bi)(a - bi)} = \frac{1}{a^2 + b^2}(a - bi).$$

- The *complex conjugate* of $z = a + bi$ is denoted by $\bar{z} = a - bi$, the *norm* of $z = a + bi$ is defined by $|z| = \sqrt{a^2 + b^2}$, hence

$$z \cdot \bar{z} = |z|^2.$$

We may also represent a complex number using polar coordinates:

$$z = a + bi = r(\cos\theta + i\sin\theta),$$

where $r = \sqrt{a^2 + b^2}$, $\theta = \arctan(b/a)$. We define $\arg(z) = \theta$.

Polar representation is useful when multiplying complex numbers. Suppose $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$\begin{aligned}
z_1 \cdot z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
&= r_1 r_2\Big((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\Big) \\
&= r_1 r_2\Big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\Big).
\end{aligned}$$

That is, we have

$$\arg(z_1 \cdot z_2) = \arg(z_1) \cdot \arg(z_2).$$

Theorem 1. $\cos\theta + i\sin\theta = e^{i\theta}$.

When $\theta = \pi$, then we have the famous Euler's formula:

$$e^{i\pi} + 1 = 0.$$

The number $e \approx 2.7182818\cdots$ is an irrational number defined through the limit

$$e = \lim_{N\to\infty}\left(1 + \frac{1}{N}\right)^N.$$

One can also show that

$$\begin{aligned} e^x &= \left(\lim_{N\to\infty}\left(1+\frac{1}{N}\right)\right)^x \\ &= \lim_{N\to\infty}\left(1+\frac{1}{N}\right)^{Nx} \text{ by Monotone convergence theorem,} \\ &= \lim_{n\to\infty}\left(1+\frac{x}{N}\right)^{N} \text{ by Bernoulli inequality.} \end{aligned}$$

Hence we have

$$e^{i\theta} = \lim_{N\to\infty}\left(1+\frac{i\theta}{N}\right)^N.$$

As $N \to \infty$, small angle approximation yields

$$1 + i\frac{\theta}{N} \approx \cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{N}\right),$$

$$\left(1 + i\frac{\theta}{N}\right)^N \approx \left(\cos\left(\frac{\theta}{N}\right) + i\sin\left(\frac{\theta}{N}\right)\right)^N.$$

1.2 Fundamental theorem of algebra

Theorem 2. *If $f(z) = a_n z^n + \cdots + a_1 z + a_0$ is a polynomial with complex coefficients, then $\exists z \in \mathbb{C}$ such that $f(z) = 0$.*

First proof. If $|z| >> 0, |z| = R > 1$, then $|a_n z^n| = |a_n| \cdot R^n$,

$$\begin{aligned} |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| &\leq |a_{n-1}| \cdot R^{n-1} + \cdots + |a_1| \cdot R + |a_0| \\ &\leq R^{n-1} \cdot (|a_0| + |a_1| + \cdots + |a_{n-1}|). \end{aligned}$$

Let z_0 be a point inside the open disc $|z| = R$ such that $|f(z_0)|$ is minimized. By transforming z_0 to 0, we may assume $|f(0)|$ is the minimum, and $|f(0)| = |a_0|$. Now suppose after transformation, the polynomial is

$$f(z) = a_0 + a_k z^k + \cdots + a_n z^n = a_0 + a_k z^k (1 + a_{k+1} z + \cdots + a_n z^{n-k})$$

where a_k is the first non-zero coefficients. Choose $z = \left(-\frac{a_0}{a_k}\right)^{1/k} \epsilon$ for some small $\epsilon > 0$, then

$$\begin{aligned} f(z) &= a_0 - a_0 \epsilon^k \left(1 + \frac{a_{k+1}}{a_k} z + \cdots + \frac{a_n}{a_k} z^{n-k}\right) \\ &\approx a_0(1 - \epsilon k), \end{aligned}$$

which implies $|f(z)| < |f(0)|$, a contradiction to the minimality of $|f(0)|$, unless $a_0 = 0$. Hence in original polynomial $f(z_0) = 0$, which is a root. □

Topological proof. Wlog, suppose $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. When $|z|$ is large, $f(z)$ is dominated by z^n, and

$$|f(z) - z^n| = |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| < C \cdot |z|^{n-1},$$

where C is some constant not depending on z. Let $|z| = R$ be the circle with radius R, then z^n is the circle with radius R^n and winding n times. The remaining term $a_{n-1}z^{n-1} + \cdots + a_0$ will perturbate around $z^n = R^n$. When $R = 0$, the path is concentrated around a_0. For some R, the path must cross 0, indicating a root. $\square$

Using Liouville's theorem. We claim that any bounded differentiable function $f : \mathbb{C} \to \mathbb{C}$ is constant (Liouville's theorem). Suppose $p(z) \in \mathbb{C}[z]$ is a polynomial with no roots, then let $f(z) = \frac{1}{p(z)}$, then

$$\frac{d}{dz}\frac{1}{p(z)} = -\frac{p'(z)}{p^2(z)},$$

and by Liouville's theorem, $f(z)$ is a constant, so is $p(z)$. $\square$

1.3 Topological structures of $\mathbb{C}$

Definition 2. *A sequence of complex numbers $\{z_n\} \subset \mathbb{C}$ is said to converge to $z \in \mathbb{C}$, if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, such that $d(z_n, z) < \epsilon$ for all $n \geq N$.*

The distance is the norm of the difference: $d(z_n, z) = |z_n - z|$, since $\mathbb{C}$ is a metric space.

Definition 3. *A sequence $\{z_n\} \subset \mathbb{C}$ is Cauchy, if $\forall \epsilon > 0, \exists N \subset \mathbb{N}$, such that $|z_k - z_l| < \epsilon, \forall k, l \geq \mathbb{N}$.*

Note that $\mathbb{C}$ is complete, meaning every Cauchy sequence in $\mathbb{C}$ converges.

Definition 4. *We define the following sets in $\mathbb{C}$:*

- *The open disc with radius r centered at z_0: $D_r(z_0) := \{z \in \mathbb{C}, |z - z_0| < r\}$.*
- *The closed disc with radius r centered at z_0: $\bar{D}_r(z_0) := \{z \in \mathbb{C}, |z - z_0| \leq r\}$.*
- *The boundary of a closed disc: $C_r(z_0) := \{z \in \mathbb{C}, |z - z_0| = r\}$.*

For special notation, the unit disc is denoted by $\mathbb{D} := D_1(0)$.

Definition 5. *Given $\Omega \subset \mathbb{C}$, a point $z_0 \in \Omega$ is an interior point if $\exists r > 0$, such that $D_r(z_0) \subset \Omega$. A set $\Omega \subset \mathbb{C}$ is open, if $\forall z \in \Omega$, z is an interior point. A set $\Omega \subset \mathbb{C}$ is closed if its compliment, $\bar{\Omega}$, is open.*

The collection of all open sets in $\mathbb{C}$, $\{\mathcal{U}_i\}$ forms a *topology* on $\mathbb{C}$, i.e.,

- $\varnothing, \mathbb{C}$ are open;
- $\displaystyle\bigcap_{i=1}^{n} \mathcal{U}_i$ is open. We say it is closed under finite intersection;
- $\displaystyle\bigcup_{\mathcal{I}} \mathcal{U}_i$ is open. We say it is closed under arbitrary union.

Definition 6. *A point $z \in \mathbb{C}$ is a limit point for $\Omega \subset \mathbb{C}$, if there exists a sequence $\{z_n\} \subset \Omega$, and*

$$\lim_{n \to \infty} z_n = z.$$

Proposition 1. *If Ω is closed, then every limit point of Ω lies in Ω.*

Proof. Let $z \in \mathbb{C}, z = \lim_{n\to\infty} z_n$ where $z_n \in \Omega$. Suppose $z \notin \Omega$, then $z \in \bar{\Omega}$ open, so $\exists r > 0$, such that $D_r(z) \subset \bar{\Omega}$. But choose $\epsilon < r$, we get a contradiction. $\square$

Definition 7. *Given $\Omega \subset \mathbb{C}$, define $Int(\Omega)$ as the set of interior points of Ω, $\bar{\Omega}$ is defined as the closure of Ω, which is the set of limit points of Ω. We have*

$$Int(\Omega) \subset \Omega \subset \bar{\Omega}.$$

Finally, the boundary of Ω, denote as $\partial\Omega$, is defined by $\partial\Omega := \bar{\Omega} - Int(\Omega)$.

Definition 8. *A set $\Omega \subset \mathbb{C}$ is bounded, if $\exists R > 0$, such that $\Omega \subset D_R(0)$.*

Definition 9. *A set $\Omega \subset \mathbb{C}$ is compact, if it is both closed and bounded.*

Here definition 9 is not rigorous. The formal definition of compactness says that a set is compact if every open cover has a finite subcover. We use the Heine-Borel theorem to conclude $\Omega \subset \mathbb{R}^n$ is compact iff it is closed and bounded, and use $\mathbb{R}^2 \simeq \mathbb{C}$.

Definition 10. *A set $\Omega \subset \mathbb{C}$ is sequentially compact, if every sequence $\{z_n\} \subset \mathbb{C}$ has a convergent subsequence.*

Proposition 2. *If a set $\Omega \subset \mathbb{C}$ is compact, then Ω is sequentially compact.*

Proof. Since Ω is compact, every open cover has a finite subcover. For each $k \geq 1, k \in \mathbb{N}$, consider the families of open covers defined by $\{B_{2^{-k}}(w), w \in \Omega\}$. Then, for each k, we have a finite collection of complex numbers $w_{k_1}, \cdots, w_{k_{m_k}}$ such that

$$\Omega \subset \bigcup_{i=1}^{m_k} B_{2^{-k}}(w_{k_i}).$$

Let $\{z_n\} \subset \Omega$ be any sequence. When $k = 1$, Ω is covered by a finitely many balls of radius $1/2$. At least one of those balls contains infinitely many terms in $\{z_n\}$, so let $z_{n_1}^{(1)} \subset z_n$ be a subsequence such that $z_{n_1}^{(1)} \in B_{1/2}(\cdot)$, i.e., all terms in $\{z_{n_1}^{(1)}\}$ lie in the same ball with radius $1/2$. Using the induction idea, suppose we have $\{z_{n_k}^{(k)}\}$ chosen, then we can construct a subsequence $z_{n_{k+1}}^{(k+1)} \subset z_{n_k}^{(k)}$, such that all terms of $z_{n_{k+1}}^{(k+1)}$ lie in the same ball with radius $2^{-(k+1)}$. Now define $y_k = z_k^{(k)}$, by our construction for all k, when $m \geq k$ $y_m \in B_{2^{-k}}(\cdot)$. Hence

$$|y_m - y_n| < 2 \cdot 2^{-k} \to 0, \quad \forall m, n \geq k.$$

Hence we conclude $\{y_k\}$ is Cauchy. Since $\mathbb{C}$ is complete, so denote

$$\lim_{k\to\infty} y_k = y.$$

Finally since Ω is closed and bounded, $y \in \Omega$. Hence every sequence has a convergent subsequence. $\square$

Definition 11. *A set $\Omega \subset \mathbb{C}$ is connected, if Ω cannot be written as the union of open sets. $\Omega \subset \mathbb{C}$ is called a region if it is open and connected.*

2 Holomorphic functions

2.1 Cauchy-Riemann equations

Definition 12. *A function $f(z) : \mathbb{C} \to \mathbb{C}$ is holomorphic if it has a well defined derivative, i.e.,*

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \text{ exists and is well defined.}$$

Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, and suppose f is holomorphic. We compute the derivative $f'(z)$ in two ways: First, let $h \in \mathbb{R}$:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0} \frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h,y)-u(x,y)}{h} + i\lim_{h\to 0} \frac{v(x+h,y)-v(x,y)}{h} \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \qquad (2.1)
\end{aligned}$$

Second, let $h \in \mathbb{R}$:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(z+ih)-f(z)}{ih} &= \lim_{h\to 0} \frac{u(x,y+ih)+iv(x,y+ih)-u(x,y)-iv(x,y)}{ih} \\
&= \lim_{h\to 0} \frac{u(x,y+ih)-u(x,y)}{ih} + i\lim_{h\to 0} \frac{v(x,y+ih)-v(x,y)}{ih} \\
&= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \qquad (2.2)
\end{aligned}$$

Since f is holomophic, by (2.1) and (2.2), we have the *Cauchy-Riemann equation*:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}. \qquad (2.3)$$

If $f = (u,v) : \mathbb{R}^2 \to \mathbb{R}^2$, the Jacobian of f is defined as

$$\mathcal{J}_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \overset{(2.3)}{=} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

Note that if we let $z = a+bi$ and define a map T by $w \mapsto (a+bi)w$ where $w \in \mathbb{C}$. Take the basis of $\mathbb{C}$ as $(1,0),(0,i)$, then the transformation matrix is given by

$$T(w) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} w.$$

Think of $z = \cos\theta + i\sin\theta$, we hence have the transformation matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

which is the counter-clockwise rotation by θ.

Theorem 3. *If a function $f(z) : \mathbb{C} \to \mathbb{C}$ satisfies the Cauchy-Riemann equation, then $f(z)$ is holomorphic.*

Proof. Let $h = h_1 + ih_2$, $f(z) = u(x, y) + iv(x, y)$. By linear approximation, we have

$$u(x + h_1, y + h_2) = u(x, y) + \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 + ||h|| \cdot \psi_1(h), \tag{2.4}$$

where $\psi_1(h) \to 0$ as $h \to 0$. Likewise,

$$v(x + h_1, y + h_2) = v(x, y) + \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 + ||h|| \cdot \psi_2(h), \tag{2.5}$$

where $\psi_2(h) \to 0$ as $h \to 0$. Then combine (2.4) and (2.5) and use Cauchy-Riemann equation, we have

$$f(z + h) = f(z) + \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) (h_1 + ih_2) + ||h|| \cdot \psi(h),$$

where $\psi(h) \to 0$ as $h \to 0$, hence $f'(z)$ exists. $\square$

2.2 Examples of holomorphic functions

We investigate some holomorphic functions:

① $f(z) = z^n$. We have that

$$\begin{aligned}
\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{(z+h)^n - z^n}{h} \\
&= \lim_{h \to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} z^{n-k} h^k}{h} \\
&= \lim_{h \to 0} n z^{n-1} + \binom{n}{2} h z^{n-2} + \cdots + h^{n-1} \\
&= n z^{n-1}.
\end{aligned}$$

We have $f'(z) = nz^{n-1}$.

② If $f(z), g(z)$ are holomorphic, then so is

$$\lambda_1 f(z) + \lambda_2 g(z), \quad \lambda_1, \lambda_2 \in \mathbb{C},$$

and

$$(\lambda_1 f(z) + \lambda_2 g(z))' = \lambda_1 f'(z) + \lambda_2 g'(z).$$

Based on this property, $f(z) \in \mathbb{C}[z]$ is holomorphic.

③ Power series: A power series is an infinite sum

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C},$$

we say the series converges at $z_0 \in \mathbb{C}$ if

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n z_0^n \quad \text{exists.}$$

We will see later that, if the power series converges absolutely in $\Omega \subset \mathbb{C}$, then it defines a holomorphic function.

④ A function that is not holomorphic: $f(z) = \bar{z}$. Derivative differ if you approach from real and imaginary axis.

2.3 Power series

As we have shown, polynomials are holomorphic. Indeed, holomorphic functions form a ring.

Definition 13. *A power series is a formal expression of the form*

$$\sum_{n=0}^{\infty} a_n z^n, a_n \in \mathbb{C}.$$

If $z \in \mathbb{C}$, we say that the series $\sum a_n z^n$ converges if

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n z^n \text{ converges.}$$

We say $\sum a_n z^n$ converges absolutely, if

$$\sum_{n=1}^{\infty} |a_n| \cdot |z^n| \text{ converges.}$$

Theorem 4. *If $\sum a_n z^n$ convergence absolutely, then $\sum a_n z^n$ converges.*

Proof. Note that for sufficiently large N,

$$\left| \sum_{n=N}^{\infty} a_n z^n \right| \leq \sum_{n=N}^{\infty} |a_n| \cdot |z^n| < \epsilon.$$

□

Theorem 5. *Given a power series $\sum a_n z^n$, $\exists 0 \leq R \leq \infty$, such that the following properties hold:*
(i) If $|z| < R$, then $\sum a_n z^n$ converges absolutely;
(ii) If $|z| > R$, then $\sum a_n z^n$ diverges.
Furthermore, $L = \frac{1}{R} = \limsup |a_n|^{1/n}$. R is called the radius of convergence.

Proof. Let $L = \frac{1}{R}$, suppose $|z| < R$, then $\exists \epsilon > 0$, such that $r := (L + \epsilon) \cdot |z| < LR < 1$, and $\exists N$ such that $L + \epsilon > |a_n|^{1/n}$, $\forall n \geq N$. Hence

$$|a_n|^{1/n} \cdot |z| < (L + \epsilon) \cdot |z| < 1 \implies |a_n| \cdot |z|^n < r^n.$$

Hence $\sum |a_n| \cdot |z|^n$ converges by comparing test with $\sum r^n, r < 1$. On the other hand, if $|z| > R$, then there are infinitely many n for which $|a_n|^{1/n} > \frac{1}{R} - \epsilon$, so

$$|a_n|^{1/n} \cdot |z| > 1 \implies |a_n| \cdot |z|^n > r^n,$$

and $\sum |a_n| \cdot |z|^n$ diverges by comparing test. □

Example 1. *Consider the radius of convergence for the following series:*

(i) $\sum n! z^n$, $R = 0$.
(ii) $\sum \frac{z^n}{n!}$, $R = \infty$.
(iii) $\sum z^n$, $R = 1$.

Theorem 6. *A power series $f(z) = \sum a_n z^n$ admits a derivative on its disc of convergence, and $f'(z) = \sum n a_n z^{n-1}$ with the same radius of convergence.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n, g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$, and write

$$f(z) = \underbrace{\sum_{n=0}^{N} a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N+1}^{\infty} a_n z^n}_{E_N(z)}.$$

Fix $z_0 \in D_R(0)$,

$$\begin{aligned}\left|\frac{f(z_0) + h) - f(z_0)}{h} - g(z_0)\right| &= \left|\frac{S_N(z_0 + h) - S_N(z_0)}{h} - S_N'(z_0) + S_N'(z_0) - g(z_0) + \frac{E_N(z_0 + h) - E_N(z_0)}{h}\right| \\ &\leq |A| + |B| + |C|,\end{aligned}$$

where

$$|A| = \left|\frac{S_N(z_0 + h) - S_N(z_0)}{h} - S_N'(z_0)\right|,$$

$$|B| = |S_N'(z_0) - g(z_0)|, \quad |C| = \left|\frac{E_N(z_0 + h) - E_N(z_0)}{h}\right|.$$

Note that $\forall \epsilon > 0, \exists N_1$, such that

$$|S_N'(z_0) - g(z_0)| = \left|\sum_{n=N}^{\infty} n a_n z^n\right| < \epsilon, \forall N \geq N_1, \tag{2.6}$$

and moreover

$$\begin{aligned}\left|\frac{E_N(z_0 + h) - E_N(z_0)}{h}\right| &\leq \sum_{n=N+1}^{\infty} |a_n| \cdot \frac{(z_0 + h)^n - z_0^n}{h} \\ &= \sum_{n=N+1}^{\infty} |a_n| \cdot \frac{h((z_0 + h)^{n-1} + z_0(z_0 + h)^{n-2} + \cdots + z_0^{n-1})}{h} \\ &\leq \sum_{n=N+1}^{\infty} |a_n| \cdot n \cdot R^{n-1}.\end{aligned}$$

Hence $\exists N_2$, such that

$$\left|\frac{E_N(z_0 + h) - E_N(z_0)}{h}\right| < \epsilon, \forall N \geq N_2.$$

Note that $\exists\delta$, such that $\forall|h| < \delta$, $|A| < \epsilon$ by the definition of derivative. Hence take $N \geq \max\{N_1, N_2\}$, one have

$$\left|\frac{f(z_0+h)-f(z_0)}{h} - g(z_0)\right| < 3\epsilon.$$

□

Corollary 1. $f(z) = \sum a_n z^n \in C^\infty(D_R(0))$.

More generally, we write $\sum a_n(z-z_0)^n$ converges for $z \in D_R(z_0)$.

Definition 14. *A function $f : \Omega \to \mathbb{C}$, where Ω is a region, is analytic, if it admits a power series on $D_R(z_0)$ for all $z_0 \in \Omega$.*

We shall see that, f is holomorphic $\Longleftrightarrow$ f is analytic.

3 Complex integrals

3.1 Integration along curves

Definition 15. *A parametrized curve is a function $\gamma : [0,1] \to \mathbb{C}$, where γ is differentiable with continuous non-zero derivative.*

Definition 16. *2 parametrization $\gamma_1, \gamma_2 : [0,1] \to \mathbb{C}$ are equivalent if $\exists s : [0,1] \to [0,1]$ smooth, $s(0) = 0, s(1) = 1$, such that $\gamma_2 = \gamma_1 \circ s$.*

Definition 17. *If γ is a parametrized curve, then*

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t)) \cdot \gamma'(t)dt.$$

The complex line integral has the following properties:

① Linearity: If $f, g : \mathbb{C} \to \mathbb{C}$ and γ is a curve, $a, b \in \mathbb{C}$ constant, then

$$\int_\gamma af + bg = a\int_\gamma f + b\int_\gamma g.$$

② If $\gamma : [0,1] \to \mathbb{C}$, define $\gamma^{-1}(t) = \gamma(1-t)$, then

$$\int_\gamma f(z)dz = -\int_{\gamma'} f(z)dz.$$

③ Invariance about parametrization: Suppose $\gamma, \gamma' : [0,1] \to \mathbb{C}$ are two smooth curves such that $\gamma(0) = \gamma'(0); \gamma(1) = \gamma'(1)$, then

$$\int_\gamma f(z)dz = \int_{\gamma'} f(z)dz.$$

Proof. Suppose $\gamma(t) : [0,1] \to \mathbb{C}$ and $\widetilde{\gamma}(s) : [0,1] \to \mathbb{C}$ are two parameterizations with same end points, and we assume that $\widetilde{\gamma}(s) = \gamma(t(s))$. We have that

$$\begin{aligned}
\int_{\widetilde{\gamma}} f(z)dz &= \int_0^1 f(\widetilde{\gamma}(s)) \cdot \widetilde{\gamma}'(s)ds \\
&= \int_0^1 f(\gamma(t(s)) \cdot \widetilde{\gamma}'(s)ds \\
&= \int_0^1 f(\gamma(t)) \cdot \gamma'(t)\frac{dt}{ds}ds \\
&= \int_0^1 f(\gamma(t)) \cdot \gamma'(t)dt.
\end{aligned}$$

□

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$$\left| \int_\gamma f(z)dz \right| \leq \int_\gamma |f(z)|dz.$$

Proof. We have that

$$\begin{aligned}
\left| \int_\gamma f(z)dz \right| &= \left| \int_0^1 f(\gamma(t)) \cdot \gamma'(t)dt \right| \\
&\leq \int_0^1 |f(\gamma(t))| \cdot |\gamma'(t)|dt \\
&\leq \int_0^1 \sup |f(\gamma(t))| \cdot |\gamma'(t)|dt \\
&= \sup_{z\in\gamma} |f(z)| \cdot \int_0^1 |\gamma'(t)|dt \\
&:= \sup_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma).
\end{aligned}$$

□

3.2 Cauchy's theorem

Definition 18. *A function $f : \mathbb{C} \to \mathbb{C}$ has a primitive,, if there exists a holomorphic function F, such that $f(z) = F'(z)$.*

Theorem 7 (Fundamental theorem of calculus)**.** *If $f(z)$ has a primitive, let γ be a curve such that $\gamma(1) = Q, \gamma(0) = P$, then*

$$\int_\gamma f(z)dz = F(Q) - F(P).$$

Proof.

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^1 f'(\gamma(t))\cdot\gamma'(t)dt\\ &= \int_0^1 \frac{d}{dt}F(\gamma(t))dt\\ &= F(\gamma(1)) - F(\gamma(0)).\end{aligned}$$

□

Definition 19. *A curve γ is closed, if $\gamma(0) = \gamma(1)$.*

Corollary 2. *If $f(z)$ has a primitive and γ is closed, then*

$$\int_\gamma f(z)dz = 0.$$

Proof. This proof is trivial. □

Theorem 8 (Cauchy's theorem)**.** *If γ is a closed curve contained in $\Omega \subset \mathbb{C}$, where $Int(\gamma) \subset \Omega$ as well. Suppose $f(z)$ is holomprphic on Ω, then*

$$\int_\gamma f(z)dz = 0.$$

In Cauchy's theorem, we do not assume any primitive on f! We need to establish some key ideas and definitions first. We first define piecewise smooth curves.

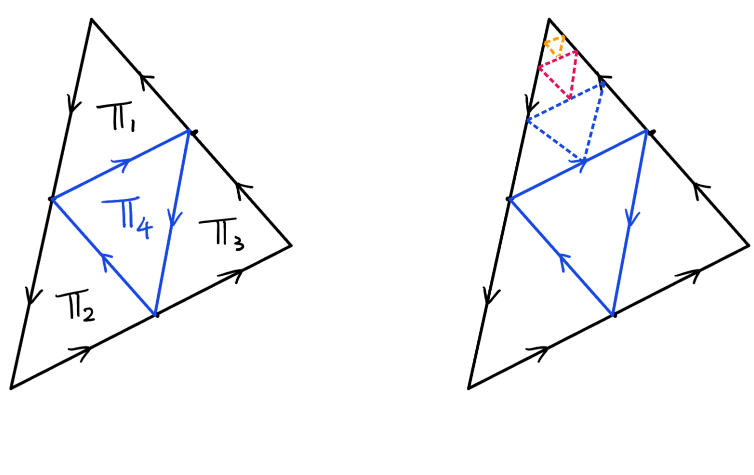
Definition 20. *A curve γ is piecewise smooth, if $\exists t_0 < t_1 < \cdots < t_n$ such that γ is smooth on each (t_j, t_{j+1}). In this case,*

$$\int_\gamma f(z)dz = \left(\int_0^{t_0} + \cdots + \int_{t_n}^1\right) f(\gamma(t))\cdot\gamma'(t)dt.$$

Lemma 1 (Goursat's lemma)**.** *Let $T \subset \mathbb{C}$ be the boundary of a closed triangle, $\mathbb{T}$ be the region bounded by T. Suppose $f(z)$ is holomorphic on $\Omega \supseteq \mathbb{T}$, then*

$$\int_T f(z)dz = 0.$$

Proof. We may decompose $\mathbb{T}$ into four parts by connecting the midpoints on each side (as in shown in the figure).



We denote T_1, T_2, T_3, T_4 as the boundaries of the new triangles and $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbb{T}_4$ be the corresponding interiors. It is easy to see that

$$\begin{aligned}\left|\int_T f(z)dz\right| &= \left|\left(\int_{T_1} + \int_{T_2} + \int_{T_3} + \int_{T_4}\right) f(z)dz\right| \\ &\leq \left|\int_{T_1} f(z)dz\right| + \left|\int_{T_2} f(z)dz\right| + \left|\int_{T_3} f(z)dz\right| + \left|\int_{T_4} f(z)dz\right| \\ &\leq 4 \cdot \left|\int_{T^{(1)}} f(z)dz\right|,\end{aligned}$$

where $T^{(1)}$ denotes the largest triangle among T_1, T_2, T_3, T_4. We may repeat this process (the right figure) and get a nested triangles $\mathbb{T} \supseteq \mathbb{T}^{(1)} \supseteq \mathbb{T}^{(2)} \supseteq \cdots$, and $\lim_{N\to\infty} \bigcap_{n=1}^{N} \mathbb{T}^{(n)} := \{z_0\} \subset \mathbb{C}$, and

$$\left|\int_T f(z)dz\right| \leq 4^n \cdot \left|\int_{T^{(n)}} f(z)dz\right|.$$

Since $f(z)$ is holomorphic at $z_0 \in \Omega$, then

$$f(z) = f(z_0) + f'(z_0) \cdot (z - z_0) + \epsilon(z)(z - z_0),$$

where $\epsilon(z) \to 0$ as $z \to z_0$. Then

$$\begin{aligned}\left|\int_{T^{(n)}} f(z)dz\right| &= \left|\int_{T^{(n)}} f(z_0) + f'(z_0) \cdot (z - z_0) + \epsilon(z)(z - z_0)dz\right| \\ &= \left|\int_{T^{(n)}} \epsilon(z)(z - z_0)dz\right|, \text{ since } f'(z) \text{ has a primitive} \\ &\leq \int_{T^{(n)}} |\epsilon(z)| \cdot |z - z_0|dz \\ &\leq \sup_z |\epsilon(z)| \cdot \int_{T^{(n)}} |z - z_0|dz \\ &\leq \epsilon_0 \cdot \int_{T^{(n)}} \mathrm{diam}(T^{(n)})dz \\ &\leq \epsilon_0 \cdot \frac{1}{2^n}\mathrm{length}(T) \cdot \frac{1}{2^n}\mathrm{diam}(T).\end{aligned}$$

Hence

$$\left|\int_{T^{(n)}} f(z)dz\right| \leq 4^{-n} \cdot \epsilon_n \cdot \mathrm{diam}(T) \cdot \mathrm{length}(T),$$

and it implies that

$$\left|\int_T f(z)dz\right| \leq \epsilon_n \cdot \mathrm{diam}(T) \cdot \mathrm{length}(T),$$

for all $n \geq 1$. $\square$

Corollary 3. *If P is a closed polygon and f is holomorphic on $\Omega \supseteq P$, then*

$$\int_P f(z)dz = 0.$$

Proof. The proof is trivial, subdivide the polygon into triangles. □

Theorem 9. *If $f(z)$ is holomorphic on a disc $D_R(z_0)$, then there exists a function $F(z)$ defined on $D_R(z_0)$ such that $F'(z) = f(z)$.*

Proof. Let $\gamma(z_0, z)$ be the straight line from z_0 to z, consider

$$F(z) := \int_{\gamma(z_0,z)} f(w)dw,$$

then

$$\begin{aligned} F'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h\to 0} \frac{\int_{\gamma(z_0,z+h)} f(w)dw - \int_{\gamma(z_0,z)} f(w)dw}{h} \\ &= \lim_{h\to 0} \frac{\int_{\gamma(z,z+h)} f(w)dw}{h}, \text{ by Goursat's lemma.} \end{aligned}$$

Now

$$\begin{aligned} \int_{\gamma(z,z+h)} f(w)dw &= \int_{\gamma(z,z+h)} f(z) + f'(z)\cdot(z-w) + \epsilon(w)\cdot(z-w)dw \\ &= \int_{\gamma(z,z+h)} f(z) + g(w)\cdot(z-w)dw, \text{ where } g \text{ is bounded as } z \to w. \\ &= h \cdot f(z) + \int_{\gamma(z,z+h)} g(w)\cdot(z-w)dw \end{aligned}$$

$$\leq h \cdot f(z) + \int_{\gamma(z,z+h)} |g(w)|\cdot|z-w|dw$$

$$\leq h \cdot f(z) + C||h||^2.$$

Hence

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

□

Theorem 10. *If f is holomorphic in a disc Ω and γ is a closed path in Ω, then*

$$\int_\gamma f(z)dz = 0.$$

Proof. By the previous theorem, it is trivial. □

3.3 An application to Cauchy's theorem: The Fourier transform

Consider the Gaussian function

$$f(x) = e^{-\pi x^2},$$

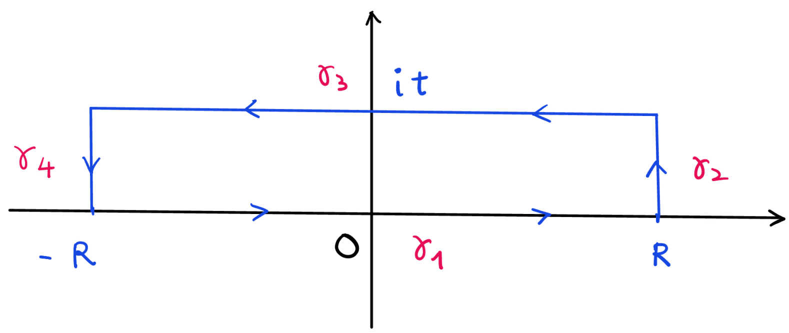
it is easy to see that $\int_{-\infty}^{\infty} f(x)dx = 1$.

Definition 21. *The Fourier transform of a function $f(x)$ is given by*

$$\widehat{f(t)} = \int_{-\infty}^{\infty} f(x) \cdot e^{-2\pi i t x} dx.$$

Theorem 11. *If $f(x) = e^{-\pi x^2}$, then $\widehat{f(t)} = e^{-\pi t^2}$.*

Proof. Consider $f(x)$ as a function of a complex variable $f(z) = e^{-\pi z^2}$, let γ_R be the following contour:



Let $\gamma_R = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$. By Cauchy's theorem,

$$\left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} \right) e^{-\pi z^2} dz = 0.$$

Note that

$$\int_{\gamma_1} f(z)dz = \int_{-R}^{R} e^{-\pi x^2} dx \to 1, \quad \text{as } R \to \infty.$$

Let $\gamma_2 : [0, t] \to \mathbb{C}, z \mapsto R + iw$, we then have

$$\begin{aligned} \left| \int_{\gamma_2} f(z)dz \right| &= \left| \int_0^t e^{-\pi (R+iw)^2} i dw \right| \\ &= \left| e^{-\pi R^2} \int_0^t e^{\pi w^2} \cdot e^{-2\pi i R w} i dw \right| \\ &\leq e^{-\pi R^2} \int_0^t e^{\pi w^2} dw, \end{aligned}$$

hence

$$\lim_{R\to\infty} \int_{\gamma_2} f(z)dz = 0,$$

and same idea applies for γ_4. Finally

$$\begin{aligned}\int_{\gamma_3} e^{-\pi z^2} dz &= \int_R^{-R} e^{-\pi(w+it)^2} dw \\ &= -e^{\pi t^2} \int_{-R}^{R} e^{-\pi w^2} \cdot e^{-2\pi i w t} dw,\end{aligned}$$

hence

$$\lim_{R\to\infty} \int_{\gamma_3} f(z)dz = -e^{\pi t^2} \cdot \widehat{f(t)}.$$

$\square$