Partial Differential Equations

Fall 2025, Math 475 Course Notes

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Contents

First-Order Linear PDEs

In this section, we will dive into some basic and elementary structures of partial differential equations. Throughout the section, consider $u(x_1, \cdots, x_n)$ as a multivariate function, $u_{x_1}, \cdots, u_{x_n}$ as its first order partial derivatives, and $u_{x_i x_j} \cdots$ as second order partial derivatives and so on.

1.1 Constant Coefficient Linear PDEs

In this part, let $u(x,y) : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, $a, b \in \mathbb{R}$ not both zero, and we are interested in the PDE of the form

$$\boxed{au_x + bu_y = 0}$$

A very intuitive example is $u(x,y) = bx - ay$, where $u_x = b, u_y = -a$ and $au_x + bu_y = ab - ab = 0$. But in general how can we solve it? We will give two methods.

1.1.1 Geometric Method Via Directional Derivative

When we have a multivariate function (in our example it suffices to illustrate with 2 variables) $z = u(x,y)$, the two partial derivatives are defined by

Definition

Definition 1. *Let $z = u(x,y)$ be a multivariate function, then the partial derivatives u_x, u_y at (x_0, y_0) are defined as*

$$u_x := \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x}$$
$$u_y := \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}.$$

Just like normal derivative with one variable, the partial derivatives u_x, u_y are the rate of change along the x, y axis respectively. We may further extend this idea to any direction, say the "partial derivative" or the "rate of change" along the line $y = x$, $y = 2x - 3$ (a linear combination of x, y coordinates) etc. That's where we introduce directional derivatives. So we may then define the directional derivative along a vector $\mathbf{v} = (a,b)$:

Definition

Definition 2. *The directional derivative of $u(x,y)$ at (x_0, y_0) along $=(a,b)$ is given by*

$$D_{\mathbf{v}} u(x_0, y_0) = \lim_{h \to 0} \frac{u(x_0 + ha, y_0 + hb) - u(x_0, y_0)}{h}$$

and we can easily see that u_x, u_y is just the special case when $\mathbf{v} = (1,0), (0,1)$ respectively. Below is a figure to illustrate directional derivative geometrically:

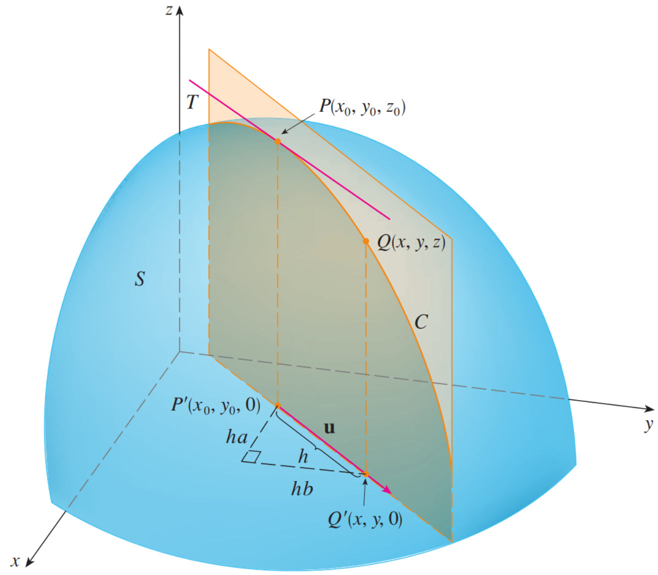

Figure 1: The directional derivative at (x_0, y_0, z_0) of the function $z = u(x, y)$. It can be viewed as the slope of tangent line of the curve obtained by slicing the function with a vertical plane that passes through the direction of the directional derivative. (*Credit: James-Stewart Calculus, early transcendentals, Eighth Edition, Page 946*)

Theorem

Theorem 1. *The directional derivative of $u(x, y)$ along $\mathbf{v} = (a, b)$ can be computed by*

$$D_{\mathbf{v}}u(x, y) = u_x(x, y) \cdot a + u_y(x, y) \cdot b$$

The proof of this theorem can be done by using Chain rule and hence the readers are encouraged to try it themselves.

After knowing some directional derivatives, we go back to the equation $a \cdot u_x + b \cdot u_y = 0$, we realized this means the directional derivative of $u(x, y)$ along $\mathbf{v} = (a, b)$ is equal to zero, meaning $u(x, y)$ is constant along $\mathbf{v}$. The line equation of $\mathbf{v}$ can be expressed by $bx - ay = 0$, and the set of lines parallel to $\mathbf{v}$ has the general form of $bx - ay = C$, where $C \in \mathbb{R}$. They are called the *characteristic lines*, and on each of those lines $D_{\mathbf{v}}u(x, y)$ is a constant and the entire plane $\mathbb{R}^2$ is generated by the set of all characteristic lines. So we see that in this case $u(x, y)$ only depends on $bx - ay$, i.e which characteristic line it belongs to, and hence the general solution to $au_x + bu_y = 0$ is

$$\boxed{u(x, y) = f(bx - ay)}$$

where f is a function of one variable, and the exact f may be obtained once extra conditions are given. To verify the correctness of (1.6), we two partial derivatives:

$$u_x = bf'(bx - ay), u_y = -af'(bx - ay)$$

and hence $au_x + bu_y = abf'(bx - ay) - abf'(bx - ay) = 0$.

Example

Example 1. *Solve the PDE* $4u_x - 3u_y = 0$ *with auxiliary condition* $u(0,y) = y^3$.

Solution 1. *It is first easy to see that* $u(x,y) = f(-3x-4y)$ *is the general solution, where* $-3x-4y = C$ *is the characteristic line. Then plugging in the auxiliary condition,* $y^3 = u(0,y) = f(-4y)$ *we get* $f(-4y) = y^3$ *and by a change of variable we see that* $f(t) = -t^3/64$. *So we have* $u(x,y) = f(-3x-4y) = (-3x-4y)^3/64$.

Example

Example 2. *Solve the PDE* $xu_x + yu_y = 0$.

Solution 2. *We notice that the directional derivatives along* (x,y) *is constant zero, and we have*

$$\frac{dy}{dx} = \frac{y}{x}$$

which brings us to a separable differential equation and we have

$$\int \frac{1}{y} dy = \int \frac{1}{x} dx + C$$

and we have $\ln|y| - \ln|x| = C$, *which is the characteristic curve of the PDE, and given any function* g *defined through* $g(\ln|y| - \ln|x|)$ *will be a solution.*

1.1.2 Change of Variable Method

Now we present a different method using change of variables. Imagine using the direction of $\mathbf{v} = (a,b)$ as our new coordinates x' and $\mathbf{v}^{\perp}$ as y'. Then we can verify that

$$x' = ax + by \qquad y' = bx - ay$$

and hence we have the relation that

$$x = \frac{ax' + by'}{a^2 + b^2} \qquad y = \frac{bx' - ay'}{a^2 + b^2}$$

and by Chain rule, we get

$$u_x = \frac{\partial u}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'} \cdot \frac{\partial y'}{\partial x} = au_{x'} + bu_{y'}$$

$$u_y = \frac{\partial u}{\partial x'} \cdot \frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'} \cdot \frac{\partial y'}{\partial y} = bu_{x'} - au_{y'}$$

so we have

$$au_x + bu_y = a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = (a^2 + b^2)u_{x'}$$

and we obtain a new equation $(a^2 + b^2)u_{x'} = 0$, which indicates the solution $u(x,y)$ is independent of x' and hence we have $u(x',y') = f(y') = f(bx - ay)$. Note that this is the exact same answer we get using the directional derivative method.

Then, we are interested in a more general form

$$\boxed{au_x + bu_y = f(x,y)}$$

for a given function $f(x,y)$. If f reduces to zero then it is the case in section 1.1.1.

Theorem

Theorem 2. *The solution to (1.13) is given by*

$$u(x,y) = \frac{1}{\sqrt{a^2+b^2}} \int_L f ds + g(bx - ay)$$

where L is the characteristic line segment from y axis to (x,y).

Proof. We will again use the change of variable method, using the change of variables discussed in (1.8), and the relations stated in (1.9);(1.10);(1.11), we have

$$a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = f\left(\frac{ax'+by'}{a^2+b^2}, \frac{bx'-ay'}{a^2+b^2}\right)$$

which simplifies to

$$u_{x'} = \frac{1}{a^2+b^2} \cdot f\left(\frac{ax'+by'}{a^2+b^2}, \frac{bx'-ay'}{a^2+b^2}\right)$$

hence integrate 1.16 with respect to x' gives

$$u(x',y') = \int_{x'} \frac{1}{a^2+b^2} \cdot f\left(\frac{ar+by'}{a^2+b^2}, \frac{br-ay'}{a^2+b^2}\right) dr + g(y')$$

and we now use the original coordinates and we have

$$u(x,y) = \int_{ax+by} \frac{1}{a^2+b^2} \cdot f\left(\frac{ar+b(bx-ay)}{a^2+b^2}, \frac{br-a(bx-ay)}{a^2+b^2}\right) dr + g(bx-ay)$$

by a simple change of variable $s = \frac{r}{\sqrt{a^2+b^2}}$, we have

$$u(x,y) = \frac{1}{\sqrt{a^2+b^2}} \int_{\frac{ax+by}{\sqrt{a^2+b^2}}} f\left(\frac{b^2x + a(\sqrt{a^2+b^2}s - by)}{a^2+b^2}, \frac{a^2y + b(\sqrt{a^2+b^2}s - ax)}{a^2+b^2}\right) ds + g(bx-ay) \tag{1.1}$$

and denoted by

$$u(x,y) = \frac{1}{\sqrt{a^2+b^2}} \int_L f ds + g(bx - ay)$$

and as we can see L is the line segment from y axis to the point (x,y).

■

Example

Example 3. *Solve the PDE $u_x + u_y = 1$.*

Solution 3. *We follow the result of theorem 2, we have $f(x,y) = 1, a = b = 1$, then we plug these values into (1.20), we have*

$$u(x,y) = \frac{1}{\sqrt{2}} \int_{\frac{x+y}{\sqrt{2}}} ds + g(x-y)$$
$$= \frac{x+y}{2} + g(x-y)$$

and we get

$$u_x = \frac{1}{2} + g'(x-y) \qquad u_y = \frac{1}{2} - g'(x-y)$$

where $g(x-y)$ is an arbitrary single variable function that depends through $x-y$, and notice that $u_x + u_y = 1$ which means the solution is valid.

Example

Example 4. *Solve the PDE $u_x + u_y = xy$.*

Solution 4. *In this example, we have $f(x,y) = xy, a = b = 1$ and (1.20) gives us*

$$\begin{aligned}
u(x,y) &= \frac{1}{\sqrt{2}} \int_{\frac{x+y}{\sqrt{2}}} \left(\frac{x+\sqrt{2}s-y}{2} \right) \cdot \left(\frac{y+\sqrt{2}s-x}{2} \right) ds + g(x-y) \\
&= \frac{1}{\sqrt{2}} \int_{\frac{x+y}{\sqrt{2}}} \frac{2s^2-(x-y)^2}{4} ds + g(x,y) \\
&= \left[\frac{2s^3-3s(x-y)^2}{12} \right] \Bigg|_{s=\frac{x+y}{\sqrt{2}}} + g(x-y) \\
&= -\frac{(x+y)(x^2+y^2-4xy)}{12} + g(x-y)
\end{aligned}$$

hence we get the general solution to the original PDE:

$$u(x,y) = -\frac{(x+y)(x^2+y^2-4xy)}{12} + g(x-y)$$

where $g(x-y)$ is an arbitrary single valued function which depends through $x-y$. To verify, note that we have

$$\begin{aligned}
u_x &= -\frac{1}{12}\Big(x^2+y^2-4xy+(x+y)(2x-4y)\Big) + g'(x-y) \\
&= \frac{1}{4}\Big(-x^2+y^2+2xy\Big) + g'(x-y) \qquad (1.2)
\end{aligned}$$

and similarly

$$\begin{aligned}
u_y &= -\frac{1}{12}\Big(x^2+y^2-4xy+(x+y)(2y-4x)\Big) - g'(x-y) \\
&= \frac{1}{4}\Big(x^2-y^2+2xy\Big) - g'(x-y) \qquad (1.3)
\end{aligned}$$

where

$$u_x + u_y = \frac{1}{4}\Big(-x^2+y^2+2xy+x^2-y^2+2xy\Big)+g'(x-y)-g'(x-y)$$
$$= xy. \tag{1.4}$$

1.2 Nonhomogeneous Problem and Method of Characteristics

We now generalize the idea of transport equation into $\mathbb{R}^n$. Suppose $x, b \in \mathbb{R}^n$, $t \in (0,\infty)$ and Du is the gradient of u with respect to x_i, then consider the problem

$$\begin{cases} u_t + b\cdot Du = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

To solve this, we inspired by the homogeneous problem, suppose we set $z(s) := u(x+sb, t+s)$ for $s \in \mathbb{R}$, then differentiating z will yield

$$\dot{z}(s) = b\cdot Du(x+sb,t+s) + u_t(x+sb,t+s) = f(x+sb,t+s)$$

then we notice that $u(x,t) = u(x+0b,t+0) = z(0)$ and $u(x-tb,t-s) = u(x-tb,0) = g(x-tb) = z(-t)$, so we have

$$\begin{aligned} u(x,t) - g(x-tb) = z(0) - z(-t) &= \int_{-t}^{0} \dot{z}(s)ds \\ &= \int_{-t}^{0} f(x+sb,t+s)ds \\ &= \int_{0}^{t} f(x+(s-t)b,s)ds \end{aligned}$$

hence, we have the general solution given by

$$u(x,t) = g(x-tb) + \int_0^t f(x+(s-t)b,s)ds$$

Example

Example 5. *Solve the PDE $u_t + u_x = 1$ with $u(x,0) = x^2$.*

Solution 5. *In this case observe that $f \equiv 1$, $g = x^2$, $Du = u_x, b = 1$, so by the formula we proposed, we have*

$$u(x,t) = (x-t)^2 + \int_0^t 1ds = x^2 - 2xt + t^2 + t.$$

It is easy to see that the function above solves the PDE.

We now study a more general form: $F(Du,u,x) = 0$ in U subject to the boundary condition $u = g$ on Γ where $\Gamma \subseteq \partial U$, $g : \Gamma \to \mathbb{R}$, also $F, g \in C^\infty$. So what is the *method of characteristics*? Well we pick $x \in U$, and then another $x_0 \in \Gamma$. Since we already know on Γ, $u = g$ so we are able to compute the value at x_0. Then we try to connect x_0, x by some nice curve u that solves the PDE.

Suppose such a curve which is described parametrically by $\mathbf{x}(s) = (x^1(s), \cdots, x^n(s))$ where $s \in \mathbb{R}$ is our parameter. We assume $u \in C^2$ solves the PDE $F(Du, u, x) = 0$ and let $z(s) = u(\mathbf{x}(s))$, as well as $\mathbf{p}(s) := Du(\mathbf{x}(s))$ where $p^i(s) = u_{x_i}(\mathbf{x}(s))$. In our set up, z gives the values along the curve, $\mathbf{p}$ gives the values of the gradient Du. Note that

$$\dot{p^i}(s) := \frac{d}{ds} p^i(s) = \frac{d}{ds} \sum_{j=1}^{n} u_{x_i x_j}(\mathbf{x}(s)) \dot{x^j}(s)$$

also we differentiate $F(Du, u, x) = 0$ with respect to x^i:

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(Du, u, x) u_{x_j x_i} + \frac{\partial F}{\partial z}(Du, u, x) u_{x_i} + \frac{\partial F}{\partial x_i}(Du, u, x) = 0$$

set

$$\dot{x}^j(s) = \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s))$$

then we evaluate (1.40) at $x = \mathbf{x}(s)$, we have

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)) u_{x_i x_j}(\mathbf{x}(s)) + \frac{\partial F}{\partial z}(\mathbf{p}(s), z(s), \mathbf{x}(s)) p^i(s) + \frac{\partial F}{\partial x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) = 0$$

so now use the expression above and (1.41), substitute back into (1,39) will give us

$$\dot{p}^i(s) = -\frac{\partial F}{\partial x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) - \frac{\partial F}{\partial z}(\mathbf{p}(s), z(s), \mathbf{x}(s)) p^i(s)$$

and finally differentiating $z = u(\mathbf{x}(s))$ will yield

$$\dot{z}(s) = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(\mathbf{x}(s)) \dot{x}^j(s) = \sum_{j=1}^{n} p^j(s) \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)).$$

Definition

Definition 3. *We define the following systems of ODEs as the characteristic equations of the PDE* $F(Du, u, x) = 0$*:*

$$\begin{cases} \dot{\mathbf{p}}(s) & = -D_x F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_z F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \mathbf{p}(s) \\ \dot{z}(s) & = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) \\ \dot{\mathbf{x}}(s) & = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \end{cases}$$

Expression above looks a little bit cursed, but let us consider a simple example:

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y)$$

for function $u(x,y)$. Then the characteristic curve will be

$$\begin{cases} \dot{x}(s) = a(x(s), y(s)) \\ \dot{y}(s) = b(x(s), y(s)) \\ \dot{z}(s) = c(x(s), y(s)) z + d(x(s), y(s)) \end{cases}$$

Example

Example 6. *Solve the PDE $-yu_x + xu_y = u$ with boundary condition $u(x,0) = g(x)$.*

Solution 6. *It is easy to see that $\dot{x}(s) = -y, \dot{y}(s) = x, \dot{z}(s) = z$. We first solve the system of ODEs with x, y:*

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases} \implies \begin{cases} x = c_2\cos(s) - c_1\sin(s) \\ y = c_1\cos(s) + c_2\sin(s) \end{cases}$$

Let $x(0) = x_0, y(0) = 0$, then $x(s) = x_0\cos(s), y(s) = x_0\sin(s)$. Furthermore solve for z we have $z(s) = c_3e^s$. With initial condition, $z(0) = g(x_0)$ so we have $z(s) = g(x_0)e^s$. Now given (x,y) we need to solve for x_0 and s such that they passes through the characteristics, we solve $x = x_0\cos(s), y = x_0\sin(s)$ and hence $x_0^2 = x^2 + y^2, s = \arctan(y/x)$ and hence we obtain

$$u(x,y) = g(x_0)e^s = g(\sqrt{x^2+y^2})e^{\arctan(y/x)}.$$

Example

Example 7. *Solve the PDE $u_x + u_y + u = e^{x+2y}$ with boundary condition $u(x,0) = 0$.*

Solution 7. *We first note that $\dot{x}(s) = 1, \dot{y}(s) = 1$ and $\dot{z}(s) = -1 + e^{x+2y}$, first solve the system of ODEs with x, y, we obtain*

$$\begin{cases} \dot{x}(s) = 1 \\ \dot{y}(s) = 1 \end{cases} \implies \begin{cases} x(s) = s + c_1 \\ y(s) = s + c_2 \end{cases}$$

for some constant $c_1, c_2 \in \mathbb{R}$. We now plug in the initial condition, we know that $x(0) = x_0, y(0) = 0$ (we make it this way so the point lie on the "boundary" where we know $u = g$), hence we have

$$x(s) = s + x_0 \qquad y(s) = s$$

with this information, we substitute x, y by s into z and have $\dot{z}(s) = -z(s) + e^{3s+x_0}$, which is an ODE that can be solved by using integrating factors. Note that

$$\mu(s) = \exp\left(\int 1ds\right) \qquad z(s) = \frac{1}{\mu(s)}\left(\int \mu(s)e^{3s+x_0}ds + C\right)$$

and we hence have

$$z(s) = \frac{1}{4}e^{x_0+3s} + \frac{C}{e^s}, C \in \mathbb{R}$$

with initial condition $z(0) = 0$, we have

$$z(s) = \frac{1}{4}e^{x_0+3s} - \frac{1}{4}e^{x_0-s}$$

we then use (1.51) to obtain x_0, s in terms of x, y, where $s = y, x_0 = x - y$, hence we have

$$u(x,y) = \frac{1}{4}(e^{x+2y} - e^{x-2y}).$$

The above two examples motivate us to further investigate (1.45) and derive a more specific formula for different type of PDEs:

Linear PDEs: The first case we consider is when $F(Du,u,x)$ is linear and homogeneous, which means it takes the form

$$b(x(s)) \cdot Du + c(x)u = 0$$

and it suggests that we have $F(p,z,x) = b(x(s)) \cdot p(s) + c(x(s))z(s)$, where we have $D_pF = b(x)$ and then by (1.45)

$$\dot{x}(s) = D_pF = b(x(s)) \qquad \dot{z}(s) = D_pF \cdot p(s) = b(x(s)) \cdot p(s)$$

also note that we have $F(p,x,z) = b(x) \cdot p(s) + c(x)z(s) = 0$ hence we further have the expression

$$\dot{z}(s) = -c(x(s))z(s)$$

so in summary we have

$$\begin{cases} \dot{x}(s) = b(x(s)) \\ \dot{z}(s) = -c(x(s))z(s) \end{cases}$$

as our parametrization and from this point we can easily see the parametrization we get in example 4 and 5.

Quasi-linear PDEs: We now consider the PDE of the form

$$F(Du,u,x) = b(x,u(x)) \cdot Du + c(x,u(x)) = 0$$

and we now have the parametrization $F(p,x,z) = b(x,z) \cdot p + c(x,z) = 0$, and from here we have $D_pF = b(x(s),z(s))$ so $\dot{x}(s) = b(x(s),z(s))$, and $\dot{z}(s) = b(x(s),z(s)) \cdot p(s)$. We finally use the fact that $b(x,z) \cdot p + c(x,z) = 0$ to conclude $\dot{z}(s) = -c(x(s),z(s))$ so im summary we have

$$\begin{cases} \dot{x}(s) = b(x(s),z(s)) \\ \dot{z}(s) = -c(x(s),z(s)) \end{cases} .$$

Example

Example 8. *Solve the PDE $u_x + u_y = u^2$ with boundary condition $u(x,0) = g$.*

Solution 8. *We first see that $\dot{x} = 1, \dot{y} = 1$ so we have the solution*

$$\begin{cases} x(s) = s + c_1 \\ y(s) = s + c_2 \end{cases}$$

with the boundary condition, we let x_0 be the point on g such that our parametrization yields $x(0) = x_0, y(0) = 0$ and hence we have

$$\begin{cases} x(s) = s + x_0 \\ y(s) = s \end{cases} .$$

Now for z we have $\dot{z} = z^2$, which now turns into a separable ODE and we have

$$\int \frac{1}{z^2} dz = \int 1 ds + C \implies -\frac{1}{z} = s + C$$

take $s=0$, and let $g(x_0)=z_0$, then we have $-\frac{1}{z_0}=C$ hence then we solve for z and we get

$$z(s)=\frac{z_0}{1-sz_0}:=\frac{g(x_0)}{1-sg(x_0)}$$

Note that in (1.63) our parametrization will yield $s=y; x_0=x-y$ so we substitute into (1.65) and we get

$$u(x,y)=\frac{g(x-y)}{1-y\cdot g(x-y)}$$

as the solution to the PDE, where g is any function depend through $x-y$. But notice that the solution only makes sense only if $1-yg(x-y)\neq 0$.

Wave Equations

2.1 Interpretations and Derivations

Suppose $U \subseteq \mathbb{R}^n$ is open, and we define a function $u(x,t) : \bar{U} \times [0,\infty) \to \mathbb{R}$ where $x \in U$ and t is the time variable.

Definition

Definition 4. *The wave equation is defined by*

$$u_{tt} - \Delta u = 0$$

where Δ is the Laplacian operator.

We first provide some physical interpretations to the wave equation. Suppose we have a smooth subregion $V \subseteq U$, and we consider the acceleration within V. The interpretation is easier for $n = 1,2$ or 3. Let $u(x,t)$ be the displacement in some direction of the point x at time $t \geq 0$, and the acceleration within V is then

$$\frac{d^2}{dt^2} \int_V u dx = \int_V u_{tt} dx$$

while the net force defined on ∂V is given by

$$-\int_{\partial V} \mathbf{F} \cdot \nu dS$$

where $\mathbf{F}$ is the force acting on V through ∂V, ν is the outward normal unit vector, then by Newton's second law,

$$\int_V u_{tt} dx = -\int_{\partial V} \mathbf{F} \cdot \nu dS$$

where we take the mass of the object to be unit mass, and further this identity obtains for all subregion V so we further have $u_{tt} = -\mathrm{div}\mathbf{F}$. Further for elastic bodies $\mathbf{F}$ is a function of the displacement gradient Du, hence we have the system

$$u_{tt} = \mathrm{div}\mathbf{F}(Du) = 0$$

and for small vibrations we use the approximation formula $\mathbf{F}(Du) = -aDu$ so we have $u_{tt} - a\Delta u = 0$ and wave equation is the case when $a = 1$.

2.2 d'Alembert's Formula

We first focus on the initial value problem for $u(x,t)$ defined on $\mathbb{R} \times (0,\infty)$. Consider the system

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}.$$

Theorem

Theorem 3. *The general solution to the wave equation $u_{tt} = c^2 u_{xx}$ takes the form*

$$u(x,t) = f(x+ct) + g(x-ct)$$

through some functions f, g.

Proof. By direct computation, we have

$$u_x = f'(x+ct) + g'(x-ct), u_{xx} = f''(x+ct) + g''(x-ct)$$

and

$$u_t = f'(x+ct) \cdot c + g'(x-ct) \cdot (-c), u_{tt} = c^2 f''(x+ct) + c^2 g''(x-ct)$$

and the results follows trivially. ■

Theorem

Theorem 4 (d'Alembert's Formula for Wave Equation)**.** *The d'Alembert's formula for wave equation $u_{tt} = u_{xx}$ is given by*

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy \quad \text{for } x \in \mathbb{R}, t \geq 0.$$

with initial condition $u(x,0) = g, u_x(x,0) = h$

Proof. (Evan's) Observe that we can rewrite the PDE as

$$u_{tt} - u_{xx} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0$$

we define v as

$$v(x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t)$$

then we have $v_t(x,t) + v_x(x,t) = 0$. This is a transport equation, where we see the directional derivative along $(1,1)$ is zero and the function is constant along this constraint. So we have a general solution defined by $v(x,t) = a(x-t)$ for any function a that defined through $x-t$ and $a(x) = v(x,0)$. Hence we finally have

$$u_t(x,t) - u_x(x,t) = a(x-t) \qquad \text{in } \mathbb{R} \times (0, \infty)$$

which appears to be another transport equation, the general solution takes the form

$$u(x,t) = \int_0^t a(x + (t-s) - s)ds + b(x+t) \tag{2.1}$$

$$= \frac{1}{2}\int_{x-t}^{x+t} a(y)dy + b(x+t) \tag{2.2}$$

with $b(x) := b(x,0)$. With initial conditions $b(x) = g(x)$ when $t = 0$, also

$$a(x) = v(x,0) = u_t(x,0) - u_x(x,0) = h(x) - g'(x)$$

we finally have

$$u(x,t)=\frac{1}{2}\int_{x-t}^{x+t}\Big(g(y)-g'(y)\Big)dy+g(x+t).$$

after simplification we have

$$u(x,t)=\frac{1}{2}[g(x+t)+g(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}h(y)dy, x\in\mathbb{R}, t\geq 0.$$

■

And d'Alembert's formula will give us the solution of the PDE equipped with the boundary value problem. If in general we have $u_{tt}-c^2u_{xx}=0$ for some $c\in\mathbb{R}$, then d'Alembert's formula will yield

$$u(x,t)=\frac{1}{2}[g(x+ct)+g(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}h(y)dy.$$

2.3 Energy Conservation

We go back to the 1D wave equation about a vibrating string:

$$u_{tt}-c^2u_{xx}=0$$

We first try to derive this formula physically (not via real math). Consider we have an infinity long string with two boundaries fixed, and now we apply a vibration to the string so it will move vertically.

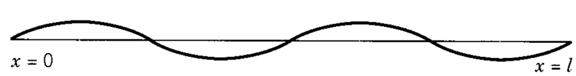

Figure 2: A vibrating string *(Source: Strauss PDE)*

At some moment, the string will look like figure 2. Let x to be a point on the string indicating its position, we denote $u(x,t)$ as the displacement from equilibrium position at time t. If we pick two points $x_0<x_1$ close enough, and we zoom in into the string and then that part of the string will look like figure 3:

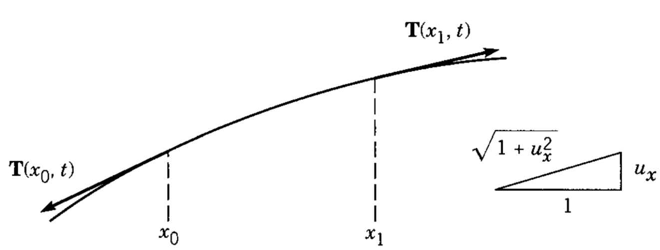

Figure 3: A zoomed vibrating string *(Source: Strauss PDE)*

We assume the string is flexible, elastic and homogeneous with density ρ, so at each point its tension vector is tangential to the string, and we further assume $\mathbf{T}$ does not depend on time t. We use $\mathbf{T}(x_1),\mathbf{T}(x_0)$ to denote the magnitude of the tension vector for the points x_0,x_1 at time t. At point a point x, u_x can be viewed as the tangent line of the string at this point, and the small triangle in figure 2 illustrates this. Since $\mathbf{T}(x)$ is parallel to the tangent line, we can then decompose the tension vector $\mathbf{T}_x$ into two directions $\mathbf{T}_x$ and $\mathbf{T}_y$. Since we assumed the string only moves vertical, so $\mathbf{T}_x = 0$. Then we have

$$\mathbf{T}_y = \mathbf{T}(x)\sin\theta = \mathbf{T}(x)\cdot\frac{u_x}{\sqrt{1+u_x^2}}$$

In segment x_0 to x_1, the net tension is simply the tension on its boundary since inside tension will cancel out, given by $\mathbf{T}(x)\sin\theta\Big|_{x=x_0}^{x=x_1}$, and at each point its mass times acceleration is just ρu_{tt}, then by integrating over the region, and apply Newton's law, we have

$$\frac{\mathbf{T}(x)u_x}{\sqrt{1+u_x^2}}\Bigg|_{x=x_0}^{x=x_1} = \int_{x_0}^{x_1}\rho u_{tt}dx$$

when the vibration is small, we immerse the philosophy from the holy engineers and write $\sqrt{1+u_x^2} = 1$, so we have

$$\mathbf{T}(x_1)u_{x_1} - \mathbf{T}(x_0)u_{x_0} = \int_{x_0}^{x_1}\rho u_{tt}dx$$

if we also assume $\mathbf{T}$ does not depend on x, differentiating the term above with respect to x will give us

$$(\mathbf{T}u_x)_x = \rho u_{tt}$$

where we let $c = \sqrt{\frac{\mathbf{T}}{\rho}}$ and hence we have

$$\boxed{u_{tt} = c^2 u_{xx}}$$

as the wave equation in one dimension.

We now talk about energy conservation in wave equation. Again for simplicity we first consider one-dimensional case.

Definition

Definition 5. *Let $u_{tt} = c^2u_{xx}$ be the wave equation of a vibrating string, we define the kinetic energy KE and potential energy PE of the string by the following:*

$$KE = \frac{1}{2}\rho\int u_t^2dx \qquad PE = \frac{1}{2}\mathbf{T}\int u_x^2dx$$

Theorem

Theorem 5. *In wave equation the total energy $E_{total} = KE + PE$ is conserved.*

Proof. We first differentiate the kinetic energy:

$$\frac{d}{dt}KE = \frac{d}{dt}\frac{1}{2}\rho\int u_t^2dx = \frac{1}{2}\rho\int\frac{d}{dt}u_t^2dx = \rho\int u_tu_{tt}dx$$

then we substitute u_{tt} with $c^2 u_{xx}$ and perform an integration by part:

$$\frac{d}{dt}KE = \rho c^2 \int u_t u_{xx} dx = \mathbf{T}\Big(u_t u_x\Big)_{x=-\infty}^{x=\infty} - \mathbf{T}\int u_x u_{tx} dx$$

where after simplifications we have

$$\frac{d}{dt}KE = -\mathbf{T}\int u_x u_{tx} dx$$

finally observe that

$$\frac{d}{dt}PE = \frac{d}{dt}\frac{1}{2}\mathbf{T}\int u_x^2 dx = \mathbf{T}\int u_x u_{xt} dx$$

it means we have

$$\frac{d}{dt}(KE + PE) = 0$$

and hence the total energy is conserved.

■

Example

Example 9 (Damped String)**.** *In a vibrating string, if there are significantly amount of air resistance, then we have the PDE*

$$u_{tt} - c^2 u_{xx} + r u_t = 0 \quad r > 0.$$

Show that the total energy decreases over time.

Solution 9. *By definition, let $c = \sqrt{\mathbf{T}/\rho}$, KE be the kinetic energy and PE be the potential energy, then*

$$\begin{aligned}
\frac{dKE}{dt} &= \frac{d}{dt}\frac{\rho}{2}\int u_t^2 dx \\
&= \rho \int u_t u_{tt} dx \\
&= \rho \int u_t (c^2 u_{xx} - r u_r) dx \\
&= \mathbf{T}\int u_t u_{xx} dx - r\rho \int u_t^2 dx \\
&= -\mathbf{T}\int u_{tx} u_x dx - r\rho \int u_t^2 dx \\
&= -\frac{d}{dt}\int \frac{\mathbf{T}}{2} u_x^2 dx - r\rho \int u_t^2 dx.
\end{aligned}$$

Hence we have

$$\frac{dE_{total}}{dt} = \frac{dKE}{dt} + \frac{dPE}{dt} = -r\rho \int u_t^2 dx < 0,$$

which means the energy is decreasing over time.

Dirac Delta Function and Distributions

3.1 Basic Definitions and Properties

Definition

Definition 6. *A distribution F is an operator $F : \phi \in C_C^\infty(\mathbb{R}) \to \mathbb{R}$ that is linear and continuous.*

For linearity, it means for all $\phi_1, \phi_2 \in C_C^\infty(\mathbb{R})$ and $c \in \mathbb{R}$, $F(\phi_1 + c\phi_2) = F(\phi_1) + cF(\phi_2)$; For continuous is means for all sequence $\{\phi_n\} \subseteq C_c^\infty(\mathbb{R})$ and $\phi \in C_c^\infty(\mathbb{R})$, if $\phi_n \to \phi$ in $C_C^\infty(\mathbb{R})$ then $F(\phi_n) \to F(\phi)$. *Note that convergence in $C_C^\infty(\mathbb{R})$ means $\phi_n^{(k)} \to \phi^{(k)}$ uniformly for all derivatives $k = 0, 1, \cdots$, and uniformly convergence means $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\sup ||\phi_n^{(k)}(x) - \phi^{(k)}(x)|| < \varepsilon$ for all x and $n \geq N$.*

For notation, we write $\langle F, \phi \rangle$ for distributions for the action of the distribution F on $\phi \in C_C^\infty(\mathbb{R})$, and then we can write the linearity as

$$\langle F, \phi_1 + c\phi_2 \rangle = \langle F, \phi_1 \rangle + c \langle F, \phi_2 \rangle$$

Definition

Definition 7. *Let $f(x)$ be a locally integrable function on $\mathbb{R}$, we define $F_f : C_C^\infty(\mathbb{R}) \to \mathbb{R}$ by*

$$\langle F_f, \phi \rangle := \int_{\mathbb{R}} f(x)\phi(x)dx$$

for all $\phi(x) \in C_C^\infty(\mathbb{R})$.

We claim that (3.2) indeed defines a distribution, we can verify its linearity and continuity easily.

Definition

Definition 8. *Define δ_0 as the distribution $\langle \delta_0, \phi \rangle = \phi(0)$ for any $\phi \in C_C^\infty(\mathbb{R})$.*

Consider the Heaviside step function $h(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$, then

$$\langle F_h, \phi \rangle = \int_{\mathbb{R}} h(x)\phi(x)dx = \int_{\mathbb{R}^+} \phi(x)dx$$

is a distribution. We now consider the derivative of a distribution, for a function $f(x)$ and its derivative $f'(x)$, we can define the following distributions:

$$\langle F_f, \phi \rangle = \int_{\mathbb{R}} f(x)\phi(x)dx \qquad \langle F_{f'}, \phi \rangle = \int_{\mathbb{R}} f'(x)\phi(x)dx$$

using integration by parts, we have

$$\int_{\mathbb{R}} f'(x)\phi(x)dx = f(x)\phi(x)\Big|_{x=-\infty}^{x=\infty} - \int_{\mathbb{R}} f(x)\phi'(x)dx = -\int_{\mathbb{R}} f(x)\phi'(x)dx \tag{3.1}$$

because ϕ is compactly supported so $\phi(\infty) = \phi(-\infty) = 0$. Note that the last term in (4.5) is simply $-\langle F_f, \phi' \rangle$, so we have the following relation:

$$\langle F_{f'}, \phi \rangle = -\langle F_f, \phi' \rangle$$

which also motivates the definition of the derivative of a distribution:

Definition

Definition 9. *Let F be a distribution, then F' is another distribution defined by*

$$\langle F', \phi \rangle := -\langle F, \phi' \rangle$$

for all $\phi \in C_C^\infty(\mathbb{R})$.

Followed by this definition, we can compute the derivative of Dirac delta function:

$$\langle \delta_0', \phi \rangle = -\langle \delta_0, \phi' \rangle = -\int_{\mathbb{R}} \delta_0 \phi(x)dx := -\phi(0).$$

Example

Example 10. *Consider the function* $p(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}.$

This function has another name called $\mathbf{ReLU}(x)$.

Solution 10. *Its distribution is given by*

$$\langle F_p, \phi \rangle = \int_{\mathbb{R}} p(x)\phi(x)dx = \int_{\mathbb{R}^+} x\phi(x)dx$$

then consider $F_{p'}$, the derivative of such distribution, we have

$$\langle F_{p'}, \phi \rangle = -\langle F_p, \phi \rangle = -\int_{\mathbb{R}^+} x\phi'(x)dx.$$

Then recall the distribution of the Heaviside step function:

$$\langle F_h, \phi \rangle = \int_{\mathbb{R}^+} \phi(x)dx := x\phi(x)\Big|_{x=0}^{x=\infty} - \int_{\mathbb{R}^+} x\phi'(x)dx = -\int_{\mathbb{R}^+} x\phi'(x)dx$$

using integration by parts and the fact that $\phi(\infty) = 0$. Then compare (3.10) and (3.9) we see that

$$\langle F_{p'}, \phi \rangle = \langle F_h, \phi \rangle.$$

Further, what will happen if we consider the derivative of the Heaviside distribution, F_h'? Let's see:

$$\langle F_h, \phi \rangle = \langle F_H', \phi \rangle = -\langle F_H, \phi' \rangle = -\int_{\mathbb{R}^+} 1 \cdot \phi'(x) dx = \phi(0)$$

since again $\phi(\infty) = 0$. Then by definition, we have

$$\langle F_h, \phi \rangle = \langle \delta_0, \phi \rangle .$$

That is, the derivative of the Heaviside function in the sense of distributions is the Dirac delta function.

Example

Example 11. *Let $f(x) = |x|$, which is, non-differentiable at $x = 0$.*

Solution 11. *We will try differentiations in the sense of distributions:*

$$\langle F_f^{'}, \phi \rangle = -\langle F_f, \phi \rangle = -\int_{\mathbb{R}} |x| \phi'(x) dx = -\left(\int_{-\infty}^{0} -x\phi'(x) dx + \int_{0}^{\infty} x\phi'(x) dx \right) \tag{3.2}$$

which is,

$$\langle F_f', \phi \rangle = x\phi(x) \Big|_{x=-\infty}^{0} - \int_{-\infty}^{0} \phi(x) dx - x\phi(x) \Big|_{x=0}^{x=\infty} + \int_{0}^{\infty} \phi(x) dx$$

and after simplifications we have

$$\langle F_f', \phi \rangle = \int_{\mathbb{R}} g(x)\phi(x) dx = \langle F_g, \phi \rangle$$

where $g(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$.

Example

Example 12. *Let* $f(x) = \begin{cases} x^2 & x \geq 0 \\ 2x + 3 & x < 0 \end{cases}$, *then what is f' in the sense of distributions?*

Solution 12. *Let us compute directly from the definition:*

$$
\begin{aligned}
\langle F_f', \phi \rangle &= -\langle F_f, \phi' \rangle \\
&= -\int_{-\infty}^{0} (2x+3)\phi'(x)dx - \int_{0}^{\infty} x^2\phi'(x)dx \\
&= -\left((2x+3)\phi(x)\Big|_{x=-\infty}^{0} - \int_{-\infty}^{0} 2\phi(x)dx + x^2\phi(x)\Big|_{x=0}^{x=\infty} - \int_{0}^{\infty} 2x\phi(x)dx \right) \\
&= -\left(3\phi(0) - 2\int_{-\infty}^{0} \phi(x)dx - \int_{0}^{\infty} 2x\phi(x)dx \right) \\
&= \int_{-\infty}^{\infty} g(x)\phi(x)dx - 3\phi(0) \\
&:= \langle g, \phi \rangle - 3\phi(0) \\
&= \langle g - 3\delta_0, \phi \rangle .
\end{aligned}
$$

where $g(x) = \begin{cases} 2 & x < 0 \\ 2x & x \geq 0 \end{cases}$. *We say that the derivative of f in the sense of distributions is the distribution $g - 3\delta_0$, or $F_g - 3\delta_0$. The $3\delta_0$ is a result of the jump discontinuity in f at $x = 0$.*

Let us consider a more general case of discontinuity. Define

$$
f(x) = \begin{cases} h(x) & x < 0 \\ g(x) & x > 0 \end{cases}
$$

where $g(0) - h(0) := c$ has a jump hence $f(x)$ is discontinuous at 0. We now consider the derivative of f in sense of distributions. By direct computation, we have

$$
\begin{aligned}
\langle F_f', \phi \rangle &= -\langle F_f, \phi' \rangle \\
&= -\left(\int_{-\infty}^{0} h(x)\phi'(x)dx + \int_{0}^{\infty} g(x)\phi'(x)dx \right) \\
&= -\left(h(x)\phi(x)\Big|_{-\infty}^{0} - \int_{-\infty}^{0} h'(x)\phi(x)dx - g(x)\phi(x)\Big|_{0}^{\infty} - \int_{0}^{\infty} g'(x)\phi(x)dx \right)
\end{aligned}
$$

where further simplification shows that

$$
\begin{aligned}
\langle F_f', \phi \rangle &= (g(0) - h(0))\phi(0) + \int_{-\infty}^{0} h'(x)\phi(x)dx + \int_{0}^{\infty} g'(x)\phi(x)dx \\
&:= \langle c\delta_0, \phi \rangle + \langle F_f', \phi \rangle
\end{aligned}
$$

where $f'(x) = \begin{cases} h'(x) & x < 0 \\ g'(x) & x > 0 \end{cases}$, and $c\delta_0$ just represents that "jump discontinuity". This idea can be generalized by the following lemma:

Lemma 1. *Suppose a function $f(x)$ is continuously differentiable except at a jump discontinuity x_0, we have $a = \lim_{x \to x_0^-} f(x)$ and $b = \lim_{x \to x_0^+} f(x)$, then*

$$f'(x) = g'(x) + (b - a)\delta_{x_0}$$

in the sense of distributions, where $g'(x)$ is the derivative of f except at x_0.

Example

Example 13. *Let*

$$f(x) = \begin{cases} 0 & x \le 0 \\ e^x & 0 < x \le 1 \\ x & 1 < x \le 2 \\ 0 & x > 2 \end{cases}.$$

Find $f'(x)$ in the sense of distributions.

Solution 13. *Use the lemma above, we can just express it as $f'(x)$ along with the jump discontinuities. Note that we have a jump up of 1 from 0^- to 0^+; a jump down of $e - 1$ from 1^- to 1^+; and a jump down of 2 from 2^- to 2^+, so we have*

$$\langle F_{f'}, \phi \rangle = \langle \delta_0 - (e-1)\delta_1 - 2\delta_2 + g(x), \phi(x) \rangle,$$

where

$$g(x) = \begin{cases} e^x & 0 < x \le 1 \\ 1 & 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}.$$

3.2 Convergence of Functions in Distributions

Definition

Definition 10. *A sequence F_n of distributions converges to a distribution F if*

$$\lim_{n \to \infty} \langle F_n, \phi \rangle = \langle F, \phi \rangle \quad \text{for all } \phi \in C_C^\infty(\mathbb{R}).$$

We may consider a sequence of locally integrable functions $\{f_n\}$, and we can define its convergence in the sense of distribution:

Definition

Definition 11. *A sequence of locally integrable functions $\{f_n\}$ converges to f in the sense of distributions if*

$$\lim_{n \to \infty} \langle F_{f_n}, \phi \rangle = \langle F_f, \phi \rangle \quad \text{for all } \phi \in C_C^\infty(\mathbb{R}),$$

which is,

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n(x)\phi(x)dx = \int_{\mathbb{R}} f(x)\phi(x)dx \quad \text{for all } \phi \in C_C^\infty(\mathbb{R}).$$

Proposition

Proposition 1. *Let $\{F_n\}, F$ be distributions, if $F_n \to F$, then $F_n^{(\alpha)} \to F^{(\alpha)}$ for all higher order derivatives.*

The proof is easy since $\phi(x) \in C_C^\infty(\mathbb{R})$.

Proposition

Proposition 2. *Let $\{f_n(x)\} \subseteq L^1_{loc}$ be a sequence of locally integrable functions and $f_n \to f$ pointwise where $f \in L^1_{loc}$. Further suppose $\exists g \in L^1_{loc}$ such that $|f_n(x)| \leq g(x)$ for all n and $x \in \mathbb{R}$, then $f_n \to f$ in the sense of distributions.*

The proof follows from monotone convergence theorem and dominated convergence theorem. An application to the proposition above is to consider the sequence

$$f_n(x) = \frac{1}{n} e^{-x^2/4n}$$

and it can be shown that $f_n(x) \to 0$ pointwise, so $f_n(x) \to 0$ in the sense of distributions.

Definition

Definition 12. *A sequence $\{f_n\} \subseteq L^1_{loc}$ converges in the sense of distributions to δ_0 (the Dirac delta function), if*

$$\int_{\mathbb{R}} f_n(x)\phi(x)dx \overset{n\to\infty}{\to} \phi(0) \quad \text{for all } \phi \in C_C^\infty(\mathbb{R}).$$

Proposition

Proposition 3. *If a sequence $\{f_n\} \subseteq L^1_{loc}$ converges to δ_0 in distribution, then $f_n(x)$ satisfies the following properties:*

(1) ***Nonnegativity:*** *$f_n(x) \geq 0$ for all $x \in \mathbb{R}, n \in \mathbb{N}$;*

(2) ***Unit mass:*** *We have $\int_{\mathbb{R}} f_n(x)dx = 1$ for all $n \in \mathbb{N}$.*

(3) ***Concentration at 0:*** *For any $\varepsilon > 0$, $f_n(x) \to 0$ uniformly on $|x| \geq \varepsilon$.*

Theorem

Theorem 6. *Let $f(x)$ be any non-negative integrable function on $\mathbb{R}$ and $\int_{\mathbb{R}} f(x)dx = 1$. Define $f_n := nf(nx)$, then $f_n \to \delta_0$ in distribution.*

Proof. We need to show $\langle f_n, \phi \rangle \to \langle \delta_0, \phi \rangle := \delta_0$, i.e $\int_{\mathbb{R}} f_n\phi \to \phi(0)$ in distribution. distribution. Using

$\varepsilon - \delta$ definition, $\forall \varepsilon > 0$, since f_n is non-negative, we have

$$\begin{aligned}
\left| \int_{\mathbb{R}} f_n(x)\phi(x)dx - \phi(0) \right| &= \left| \int_{\mathbb{R}} f_n(x) \cdot (\phi(x) - \phi(0))dx \right| \\
&\leq \int_{\mathbb{R}} f_n(x) \cdot \Big| \phi(x) - \phi(0) \Big| dx \\
&= \int_{\mathbb{R}} f(y) \cdot \left| \phi\left(\frac{y}{n}\right) - \phi(0) \right| dy
\end{aligned}$$

By continuity of ϕ, $\forall \varepsilon > 0$, one can choose δ, such that for $N \in \mathbb{N}$ sufficiently large, $\left|\frac{y}{n} - 0\right| < \delta$ for all $n \geq N$ would imply $\left|\phi\left(\frac{y}{n}\right) - \phi(0)\right| < \varepsilon$, hence

$$\int_{\mathbb{R}} f(y) \cdot \left| \phi\left(\frac{y}{n}\right) - \phi(0) \right| dy \leq \varepsilon \int_{\mathbb{R}} f(y)dy \equiv \varepsilon.$$

for all $n \geq N$, and hence

$$\left| \int_{\mathbb{R}} f_n(x)\phi(x) - \phi(0) \right| < \varepsilon$$

for all $n \geq N$ which means $f_n(x) \to \delta_0$ in distribution. ■

Theorem

Theorem 7. $\sin(nx) \to 0$ *in distribution.*

Proof. Note that $\phi \in C_C^\infty(\mathbb{R})$, so $\exists M \in \mathbb{R}^+$ such that $\phi(x) = 0$ for all $|x| \geq M$, hence

$$\int_{\mathbb{R}} \sin(nx)\phi(x)dx = \int_{[-M,M]} \sin(nx)\phi(x)dx.$$

Then we have

$$\begin{aligned}
\left| \int_{[-M,M]} \sin(nx)\phi(x)dx - 0 \right| &= \left| \int_{[-M,M]} \left(-\frac{d}{dx} \cdot \frac{\cos(nx)}{n} \right) \phi(x)dx \right| \\
&= \left| -\frac{\cos(nx)}{n} \cdot \phi(x) \Big|_{x=-M}^{x=M} + \int_{[-M,M]} \frac{\cos(nx)}{n} \cdot \phi'(x)dx \right| \\
&\leq \frac{1}{n} \int_{[-M,M]} \Big| \cos(nx) \cdot \phi'(x) \Big| dx \\
&\leq \frac{1}{n} \int_{[-M,M]} ||\phi'(x)||_{L^\infty} dx \\
&= \frac{2M||\phi'(x)||_{L^\infty}}{n}.
\end{aligned}$$

Simply let $\varepsilon := \frac{2M||\phi'(x)||_{L^\infty}}{n}$, choose N such that $N > \frac{2M||\phi'(x)||_{L^\infty}}{\varepsilon}$ so for all $n \geq N$, we have the original equation $< \varepsilon$. ■

Example

Example 14. *Show that the sequence of function defined by* $f_n(x) = \frac{1}{\pi} \cdot \frac{n}{n^2x^2+1}$ *converges to* δ_0 *in distribution.*

Solution 14. *By definition we wish to show that* $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ *such that*

$$\left| \int_{\mathbb{R}} f_n(x)\phi(x)dx - \phi(0) \right| < \varepsilon$$

for all $n \geq \mathbb{N}$. *By a similar argument in the proof of* ***theorem 6*** *we obtain*

$$\left| \int_{\mathbb{R}} f_n(x)\phi(x)dx - \phi(0) \right| \leq \int_{\mathbb{R}} f_n(x) \cdot \left| \phi(x) - \phi(0) \right| dx.$$

Since $\phi \in C_C^\infty(\mathbb{R})$, *so for the* ε *given,* $\exists \delta$ *such that* $|x| < \delta$ *implies* $|\phi(x) - \phi(0)| < \varepsilon$. *We then rewrite the right hand side of (3.42) as*

$$\int_{\mathbb{R}} f_n(x) \cdot \left| \phi(x) - \phi(0) \right| dx = \underbrace{\int_{[-\delta,\delta]} f_n(x) \cdot \left| \phi(x) - \phi(0) \right| dx}_{I} + \underbrace{\int_{\mathbb{R}/[-\delta,\delta]} f_n(x) \cdot \left| \phi(x) - \phi(0) \right| dx}_{J}$$

where we will investigate I, J *separately. For* I *term, use the fact that* $x \in [-\delta, \delta]$, *we have*

$$I < \varepsilon \int_{[-\delta,\delta]} f_n(x)dx \leq \varepsilon \tag{3.3}$$

since we have $\int_{\mathbb{R}} f_n(x)dx = 1$. *For* J, *we have*

$$J \leq \int_{\mathbb{R}/[-\delta,\delta]} f_n(x) \cdot \Big(|\phi(x)| + |\phi(0)| \Big) dx \tag{3.4}$$

$$\leq \int_{\mathbb{R}/[-\delta,\delta]} f_n(x) \cdot 2||\phi(x)||_{L^\infty} dx \tag{3.5}$$

$$:= \frac{4||\phi(x)||_{L^\infty}}{\pi} \int_\delta^\infty \frac{n}{n^2x^2+1} dx \tag{3.6}$$

$$= \frac{4||\phi(x)||_{L^\infty}}{\pi} \cdot \Big(\frac{\pi}{2} - \arctan(n\delta) \Big) \tag{3.7}$$

by choosing N *sufficiently large, we have* $\frac{\pi}{2} - \arctan(n\delta) < \varepsilon$ *for all* $n \geq N$. *So all combined, we have*

$$\left| \int_{\mathbb{R}} f_n(x)\phi(x) - \phi(0) \right| < 2\varepsilon$$

for any $\varepsilon > 0$ *and sufficiently large* $n > N$. *Hence* $f_n(x) \to \delta_0$ *in distribution.*

Heat Equations

4.1 Simple Diffusion Equations

In general, let $u(x,t)$ be the function of our interest, heat equation is the PDE that takes the form

$$u_t - k\Delta u = 0. \quad (k > 0)$$

We will first study the simple diffusion equation defined by

$$\begin{cases} u_t(x,t) = ku_{xx}(x,t) & \text{in } R \\ u(x,t) = g(x,t) & \text{on } \partial R \end{cases}$$

where we restrict the region R to be the rectangle defined by $R := [0,L] \times [0,T)$. It is intuitive to first present where equation (3.2) is coming from. Let us imagine inside a straight tube contains motionless liquid, and a dye is diffusing through the liquid (see the figure below).

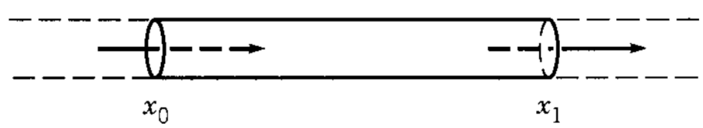

Figure 4: Illustration of our example, where the straight tube is filled with motionless liquid, where a dye is diffusing from x_0 to x_1. *(Source: Strauss PDE)*

By Fick's law the dye will move from the higher concentration region to lower concentration region, and the rate of such a motion is proportional to the concentration gradient. Suppose $u(x,t)$ is the concentration (mass per unit length) of the dye at position x at time t, then in the tube illustrated above, the total mass of the dye is given by

$$M = \int_{x_0}^{x_1} u(x,t)dx$$

then applying Fick's law, the rate of change of the concentration (dM) is "flow in from x_0 $-$ flow out from x_1" since we assume the liquid is motionless, hence we have the equation

$$\frac{dM}{dt} = k(u_x(x_0,t) - u_x(x_1,t))$$

where $k > 0$, and hence

$$\int_{x_0}^{x_1} u_t(x,t)dx = k(u_x(x_0,t) - u_x(x_1,t))$$

we differentiate with respect to x_1, and wlog it will yield

$$u_t(x,t) = ku_{xx}$$

where k is some constant.

We first study some properties of the heat equation, and will intriduce the general solution to the heat equation in the next few sections.

Theorem

Theorem 8 (Maximum Principle). *If $u(x,t)$ solves (3.2), then*

$$\max_{(u,t)\in R} u(x,t) = \max_{(u,t)\in \partial R} u(x,t)$$

The idea of the proof is that, at an interior maximum the first derivative is equal to zero and the second derivative is non-positive. If we knew $u_{xx} \neq 0$ at the maximum then $u_{xx} < 0$ as well as $u_t = 0$, which will contradicts the equation $u_t = ku_{xx}$.

Proof. Let $M := \max_{(u,t)\in\partial R} u(x,t)$. Our goal is to show $u(x,t) \leq M$ for all $(x,t) \in R$. Let $\varepsilon > 0$, let $v(x,t) = u(x,t) + \varepsilon x^2$, then it is clear that $v(x,t) \leq u(x,t) + \varepsilon L^2$ (recall $x \in [0,L]$) for all $(x,t) \in R \cup \partial R$. Further we have

$$v_t - kv_{xx} = u_t - k(u+\varepsilon x^2)_{xx} = u_t - ku_{xx} - 2\varepsilon k = -2\varepsilon k < 0$$

We now suppose $v(x,t)$ attains its maximum at an interior point (x_0,t_0), then we know $v_t = 0, v_{xx} \leq 0$ at (x_0,t_0) and it contradicts the equation above.

We then consider if $v(x,t)$ attains its maximum at $\{t_0 = T, 0 < x < L\}$, then $v_x(x_0,t_0) = 0$ and $v_{xx}(x_0,t_0) \leq 0$. Also since $v(x_0,t_0) \geq v(x_0,t_0-\delta)$ for $\delta > 0$, then

$$v_t(x_0,t_0) = \lim_{\delta\to 0} \frac{v(x_0,t_0) - v(x_0,t_0-\delta)}{\delta} \geq 0$$

(Note the maximum is only "one-sided" in the variable t), hence that's a contradiction. ■

Note that we proved the weak maximum principle, which is the maximum on the boundary is equal to the maximum inside the exterior, but there is a strong version of maximum principle which states the maximum cannot be assumed anywhere inside R but only ∂R. This is much harder to prove.

Theorem

Theorem 9. *(Uniqueness of solution) Suppose u,v both solves the PDE $\begin{cases} u_t = ku_{xx} & \text{in } R \\ u = g & \text{on } \partial R \end{cases}$, then $u = v$ for all $(x,t) \in R$.*

Proof. Let $w = u - v$, where $w_t = kw_{xx}$ in R and $w = g - g$ on ∂R. Then by maximum / minimum principle, we know that

$$\max_{(x,t)\in R} w(x,t) = \max_{(x,t)\in\partial R} w(x,t) = 0 \quad \min_{(x,t)\in R} w(x,t) = \min_{(x,t)\in\partial R} w(x,t) = 0$$

which implies $w \equiv 0$ for all $(x,t) \in R$, hence $u \equiv v$. ■

Theorem

Theorem 10 (Dirichlet Problem for the Diffusion Equation). *There is at most one solution of*

$$\begin{cases} u_t - ku_{xx} = f(x,t) & \text{for } 0 < x < l, t > 0 \\ u(x,0) = \phi(x) & \\ u(0,t) = g(t) & u(l,t) = h(t) \end{cases} \tag{4.1}$$

Proof. Let $u_1(x,t), u_2(x,t)$ that both solves (4.1), lwt $w = u_1 - u_2$, then $w_t - kw_{xx} = 0$, $w(x,0) = w(0,t) = w(l,t) = 0$. Let $T > 0$ then by maximum principle $w(x,t)$ has its maximum for the rectangle on its boundary (bottom ot sides), so we have $w(x,t) \leq 0$. The same argument using minimum principle will yield $w(x,t) \geq 0$, therefore $w(x,t) \equiv 0$ and hence $u_1 \equiv u_2$. ■

Theorem

Theorem 11 (Stability). *Suppose u, v satisfies*

$$\begin{cases} u_t = ku_{xx} & \text{in } R \\ u = g_1 & \text{on } \partial R \end{cases} \quad \text{and} \quad \begin{cases} v_t = kv_{xx} & \text{in } R \\ v = g_2 & \text{on } \partial R \end{cases}$$

then define $w := u - v$, we have

$$||w||_\infty \leq ||g_1 - g_2||_\infty.$$

Proof. By maximum principle, we know $\max_{(x,t)\in R} w(x,t) = \max_{(x,t)\in \partial R}(g_1 - g_2)$, and the result follows. ■

Proposition

Proposition 4 (Energy Method). *In $u_t = ku_{xx}$ defined on $R := [0,L] \times [0,T]$, assuming zero boundary condition ($u = 0$ at $x = 0, L$) and no flux boundary condition ($u_x = 0$ at $x = 0, L$), the energy defined by*

$$E(t) = \frac{1}{2}\int_0^L u^2(x,t)dx$$

is non-increasing.

Proof. We differentiate the energy with respect to time t and we get

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_0^L \frac{d}{dt}u^2(x,t)dx = \int_0^L u(x,t)u_t(x,t)dx$$

by substituting we get

$$
\begin{aligned}
\frac{d}{dt}E(t) &= \int_0^L u(x,t)ku_{xx}(x,t)dx && (4.2)\\
&= ku(x,t)u_x(x,t)\Big|_{x=0}^{x=L} - k\int_0^L u_x^2(x,t)dx && (4.3)\\
&= -k\int_0^L u_x^2(x,t)dx && (4.4)\\
&\leq 0 && (4.5)
\end{aligned}
$$

i.e $E'(t) \leq 0$ so the energy is non-increasing. ■

4.2 Fundamental Solutions

In this section we will solve the heat equation on the real line $\mathbb{R}$, that is, $x \in \mathbb{R}, t \in [0,\infty)$:

$$
\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = \phi(x) \end{cases} \tag{4.6}
$$

To solve (4.6), we will first introduce some invariant properties:

(i) (4.6) satisfies linearity. if u, v both solves (4.6), then so is $cu + dv$ for any scalar $c, d \in \mathbb{R}$;

(ii) (4.6) is invariant under translation. If $u(x,t)$ is a solution, then for a fixed y, $u(x-y,t)$ is also a solution;

(iii) If $u(x,t)$ solves (4.6), then any of its derivatives (u_x, u_t, u_{xx}, etc.) also solves (4.6);

(iv) Suppose $u(x,t)$ solves (4.6), then so is $u(\lambda x, \lambda^2 t)$ for any $\lambda \in \mathbb{R}$.

(v) An integral solution is again a solution. If $S(x,t)$ is a solution then so is $S(x-y,t)$ and so is

$$
v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy
$$

for any function $g(y)$.

Theorem

Theorem 12 (Fundamental Solution)**.** *The function*

$$
\Phi(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}}
$$

is the fundamental solution to heat equation $u_t = ku_{xx}$ when $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, the solution given by the boundary condition $u_t = ku_{xx}, u(x,0) = \phi(x)$ is then

$$
u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t)\phi(y)dy = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot \phi(y)dy
$$

Example

Example 15. *Find the solution $u(x,t)$ to the heat equation $u_t = ku_{xx}$ with boundary conditions $\phi(x) = 1, \forall |x| < l$ and $\phi(x) = 0$ otherwise.*

Solution 15. *We plug in the formula and we have*

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot \mathbf{1}_{|y|<l} dy = \frac{1}{\sqrt{4\pi kt}} \int_{-l}^{l} e^{-\frac{(x-y)^2}{4kt}} dy$$

let $u = \frac{x-y}{\sqrt{4kt}}$, then $du = -\frac{1}{\sqrt{4kt}} dy$ hence

$$\frac{1}{\sqrt{4\pi kt}} \int_{-l}^{l} e^{-\frac{(x-y)^2}{4kt}} dy = \frac{1}{\sqrt{4\pi kt}} \int_{-\frac{x-l}{\sqrt{4kt}}}^{\frac{x+l}{\sqrt{4kt}}} -(\sqrt{4kt}) e^{-u^2} du$$

Hence we have

$$u(x,t) = -\frac{1}{\sqrt{\pi}} \int_{-\frac{x-l}{\sqrt{4kt}}}^{\frac{x+l}{\sqrt{4kt}}} e^{-u^2} du = -\frac{1}{2} erf\left(\frac{x-l}{\sqrt{4kt}}\right) + \frac{1}{2} erf\left(\frac{x+l}{\sqrt{4kt}}\right).$$

where we define

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp.$$

Example

Example 16. *Find the solution $u(x,t)$ to the heat equation $u_t = ku_{xx}$ with boundary condition $\phi(x) = 1, x > 0$, $\phi(x) = 3, x < 0$.*

Solution 16. *We have that*

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} e^{-\frac{(x-y)^2}{4kt}} dy + \frac{3}{\sqrt{4\pi kt}} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4kt}} dy$$

where we do a similar change of variable and let $u = \frac{x-y}{\sqrt{4kt}}$, so we have

$$du = -\frac{1}{\sqrt{4kt}} dy$$

and hence we have

$$\begin{aligned} u(x,t) &= -\frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-u^2} du - \frac{3}{\sqrt{\pi}} \int_{\infty}^{\frac{x}{\sqrt{4kt}}} e^{-u^2} du && (4.7) \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du + \frac{2}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-u^2} du && (4.8) \\ &= 1 + \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-u^2} du && (4.9) \\ &= 2 - erf\left(\frac{x}{\sqrt{4kt}}\right). && (4.10) \end{aligned}$$

where we defined

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp.$$

Proposition

Proposition 5. *Any piecewise constant initial conditions on intervals can be written as a linear combinations of step functions, for example:*

$$\mathbf{1}_{(a,b)} = H(x-a) - H(x-b),$$

with this as the initial condition, the solution to the hear equation is given by

$$u(x,t) = \frac{1}{2}\left[erf\left(\frac{x-a}{\sqrt{4kt}}\right) - erf\left(\frac{x-b}{\sqrt{4kt}}\right)\right]$$

We next study some properties of the solution formula

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y) dy$$

Proposition

Proposition 6. *The fundamental solution $\Phi(x,t)$ satisfies*

$$\int_{\infty}^{\infty} \Phi(x,t) dx = 1.$$

Proof. The result is obvious using change of variables. ∎

Proposition

Proposition 7 (Maximum Principle)**.** *For the heat equation with boundary condition $u_t = ku_{xx}$ and $u(x,0) = \phi(x)$, if there exists a constant B such that $|\phi(x)| \leq B$ for all $x \in \mathbb{R}$, then $|u(x,t)| \leq B$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.*

Proof. We have

$$|u(x,t)| \leq \int_{-\infty}^{\infty} \Phi(x-y,t) \cdot |\phi(y)| dy \leq B \int_{-\infty}^{\infty} \Phi(x-y,t) dy = B.$$

∎

Proposition

Proposition 8 (Decay)**.** *Suppose $C := \int_{-\infty}^{\infty} |\phi(y)| dy < \infty$, then $\lim_{t\to\infty} |u(x,t)| = 0$.*

Proof. Note that for all $t > 0$ and x, y, we have

$$\begin{aligned}|u(x,t)| &\leq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} |\phi(y)| dy \\ &\leq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} |\phi(y)| dy \\ &= \frac{C}{\sqrt{4\pi kt}}.\end{aligned}$$

■

Theorem

Theorem 13. *Let ϕ be a bounded, continuous and integrable function on $\mathbb{R}$, define*

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y) dy$$

then $u(x,t) \in C^\infty$, and $\lim_{t\to 0} u(x,t) = \phi(x)$. Furthermore, $u(x,t)$ solves the heat equation $u_t = ku_{xx}$.

Proof. We will only proof the convergence here. Note that

$$\left| \int_{-\infty}^{\infty} \Phi(x-y,t)\phi(y)dy - \phi(x) \right| \leq \int_{-\infty}^{\infty} \Phi(x-y,t) \cdot |g(y) - g(x)| dy$$

Since ϕ is continuous and bounded, so $\exists B$ such that $|\phi(x)| \leq B$, also $\forall \varepsilon > 0$, $\exists \varepsilon$ such that $|y-x| < \delta$ implies $|\phi(y) - \phi(x)| < \varepsilon$. Hence we have

$$\begin{aligned}\int_{-\infty}^{\infty} \Phi(x-y,t) \cdot |\phi(y) - \phi(x)| dy &= \int_{|y-x|<\delta} \Phi(x-y,t) \cdot |\phi(y) - \phi(x)| dy \\ &\quad + \int_{|y-x|\geq\delta} \Phi(x-y,t) \cdot |\phi(y) - \phi(x)| dy \\ &\leq \varepsilon \int_{|y-x|<\delta} \Phi(x-y,t) dy + \int_{|y-x|\geq\delta} \Phi(x-y,t) \cdot |\phi(y) - \phi(x)| dy \\ &\leq \varepsilon + \frac{1}{\sqrt{4\pi kt}} \int_{|y-x|\geq\delta} e^{-\frac{(x-y)^2}{4kt}} \cdot |\phi(y) - \phi(x)| dy \\ &\leq \varepsilon + \frac{2B}{\sqrt{4\pi kt}} \int_{|y-x|\geq\delta} e^{-\frac{(x-y)^2}{4kt}} dy \\ &= \varepsilon - \frac{2}{\sqrt{\pi}} \int_{|u|\geq\frac{\delta}{\sqrt{4kt}}} e^{-u^2} du.\end{aligned}$$

Note that since

$$\lim_{t\to 0} \frac{2}{\sqrt{\pi}} \int_{|u|\geq\frac{\delta}{\sqrt{4kt}}}^{\infty} e^{-u^2} du = 0$$

we then have

$$\left| \int_{-\infty}^{\infty} \Phi(x-y,t)\phi(y)dy - \phi(x) \right| < \varepsilon.$$

■

Laplace Equations

5.1 Properties of Harmonic Functions

Definition

Definition 13. *Let $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, u is said to be harmonic if $\Delta u = 0$, where*

$$\Delta u = \sum_{n=1}^{n} u_{x_i x_i}$$

In this section, we study the PDE

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

and we will mainly focus on $u : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$.

Proposition

Proposition 9 (Mean Value Property)**.** *Let u be a C^2 harmonic function on $\Omega \subseteq \mathbb{R}^3$, let $x_0 \in \Omega$, $B(x_0, r) \subseteq \Omega$, then*

$$u(x_0) = \frac{1}{4\pi r^2} \iint_{\partial B(x_0,r)} u(x) dS(x)$$

Proof. We define a function

$$\phi(r) = \frac{1}{4\pi r^2} \iint_{\partial B(x_0,r)} u(x) dS(x)$$

since $u \in C^2$, by averaging lemma

$$\lim_{r \to 0} \phi(r) = \lim_{r \to 0} \frac{1}{4\pi r^2} \iint_{\partial B(x_0,r)} u(x) dS(x) = u(x_0)$$

and we now show $\phi(r)$ is constant in Ω by taking the derivative. Let $y = \frac{x - x_0}{r}$, then $dS(y) = \frac{1}{r^2} dS(x)$ hence we have

$$\phi(r) = \lim_{r \to 0} \frac{1}{4\pi r^2} \iint_{\partial B(x_0,r)} u(x) dS(x) = \frac{1}{4\pi} \iint_{\partial B(0,1)} u(x_0 + ry) dS(y)$$

now we have

$$\phi'(r) = \frac{1}{4\pi} \iint_{\partial B(0,1)} \frac{d}{dr} u(x_0 + ry) dS(y) = \frac{1}{4\pi} \iint_{\partial B(0,1)} \nabla u(x_0 + ry) y dS(y) \tag{5.1}$$

$$\text{(revert back to original coordinates)} = \frac{1}{4\pi r^2} \iint_{\partial B(x_0,r)} \nabla u(x) \cdot \frac{x - x_0}{r} dS(x) \tag{5.2}$$

and note that $\frac{x-x_0}{r}$ is the outer unit normal to $\partial B(x_0,r)$ hence by divergence theorem, we have

$$\frac{1}{4\pi r^2}\iint_{\partial B(x_0,r)} \nabla u(x)\cdot\frac{x-x_0}{r}dS(x) = \frac{1}{4\pi r^2}\iiint_{B(x_0,r)} \Delta u(x)dx \equiv 0$$

since u is harmonic. Hence $\phi(r)$ is a constant and is equal to $\phi(0)=u(x_0)$. ■

We may also averaging around a ball $B(x_0,r)\subseteq\Omega$, and we have

$$u(x_0)=\frac{1}{\frac{4}{3}\pi r^3}\iiint_{B(x_0,r)} u(x)dx.$$

Corollary

Corollary 1 (Mean Value Property in $\mathbb{R}^2$)**.** *Let u be a C^1 harmonic function on $\Omega\subseteq\mathbb{R}^2$, let $x_0\in\Omega$, $B(x_0,r)\subseteq\Omega$, then*

$$u(x_0)=\frac{1}{2\pi r}\int_{\partial B(x_0,r)} u(x)dx$$

Proof. Similarly define

$$\phi(r)=\frac{1}{2\pi r}\int_{\partial B(x_0,r)} u(x)dx \quad \phi(0)=u(x_0)$$

and by change of variable

$$y=\frac{x-x_0}{r}, \quad \phi(r)=\frac{1}{2\pi}\int_{\partial B(0,1)} u(x_0+ry)\,dy$$

now by taking derivative

$$\begin{aligned}
\phi'(r) &= \frac{1}{2\pi}\frac{d}{dr}\int_{\partial B(0,1)} u(x_0+ry)dy \\
&= \frac{1}{2\pi}\int_{\partial B(0,1)} y\nabla u(x_0+ry)dy \\
&= \frac{1}{2\pi r}\int_{\partial B(x_0,r)} \frac{x-x_0}{r}\nabla u(x)dx \\
&= \frac{1}{2\pi r}\iint_{B(x_0,r)} \Delta u(x)dx = 0.
\end{aligned}$$

■

Proposition

Proposition 10 (Maximum Principle)**.** *Let u be a C^2 harmonic function on a bounded domain $\Omega\subseteq\mathbb{R}^3$ and $u\in C(\bar{\Omega})$. if u attains its maximum at a point in Ω, then u must be identically constant inside $\bar{\Omega}$*

Proof. Suppose $\exists x_0 \in \Omega$ such that $u(x_0) = \max_{x\in\bar{\Omega}} u(x)$, then by maximum principle, pick $B(x_0,r) \subseteq \Omega$, we have

$$u(x_0) = \frac{1}{\frac{4}{3}\pi r^3}\int_{B(x_0,r)} u(y)dy,$$

which implies u must be identically constant in Ω. ■

Maximal principle can also be extended to minimum principle.

Corollary

Corollary 2 (Uniqueness)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be bounded, then there exists at most one $u \in C^2(\Omega)\cap C(\bar{\Omega})$ that solves te Dirichlet problem*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Proof. Supposeu_1,u_2 both solves the equation above, let $v = u_1 - u_2$, then we know $\Delta v = 0$ in Ω and $v = g - g \equiv 0$ on $\partial\Omega$. Then using maximum and minimum principle, $v \equiv 0$ in Ω hence $u_1 \equiv u_2$. ■

Corollary

Corollary 3 (Stability)**.** *Let $\Omega \subseteq \mathbb{R}^3$ bounded, $g_1,g_2 \in C(\partial\Omega)$, $u_1,u_2 \in C^2(\Omega)\cup C(\bar{\Omega})$ and they solves*

$$\begin{cases} \Delta u_1 = 0 & \text{in } \Omega \\ u_1 = g_1 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} \Delta u_2 = 0 & \text{in } \Omega \\ u_2 = g_2 & \text{on } \partial\Omega \end{cases}$$

then

$$\max_{x\in\Omega} |u_1(x) - u_2(x)| \le \max_{x\in\partial\Omega} |g_1(x) - g_2(x)|.$$

Proof. The proof follows immediately from maximum and minimum principle. ■

Proposition

Proposition 11 (Dirichlet's Principle)**.** *Let $\Omega \subseteq \mathbb{R}^3$, define a class of functions*

$$A_h = \left\{ w \in C^2(\Omega)\cap C(\bar{\Omega}) : w = h \text{ on } \partial\Omega \right\},$$

for some function h, then define

$$E(w) = \frac{1}{2}\iiint_\Omega |\nabla w|^2 dx.$$

Let $u \in A_h$. $E(u) \le E(w)$ for all $w \in A_h$ if and only if u solves the PDE

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = h & \text{on } \partial\Omega \end{cases}.$$

Proof. ($\Longrightarrow$): First suppose u is a minimizer over all $w \in A_h$. Let $v \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $v = 0$ on $\partial\Omega$, then $u + \varepsilon v \in A_h$ for all $\varepsilon > 0$. Define

$$f(\varepsilon) = E(u + \varepsilon v) = \frac{1}{2} \iiint_\Omega |\nabla(u + \varepsilon v)|^2 dx.$$

Since f is differentiable, we would have

$$f'(\varepsilon) = \frac{1}{2} \iiint_\Omega \frac{d}{d\varepsilon}(|\nabla u|^2 + 2\varepsilon \nabla u \cdot \nabla v + \varepsilon^2 |\nabla v|^2) dx \tag{5.3}$$

$$= \frac{1}{2} \iiint_\Omega (2\nabla u \cdot \nabla v + 2\varepsilon |\nabla v|^2) dx \tag{5.4}$$

$$\tag{5.5}$$

because u is a minimizer so $f'(0) = 0$ which implies

$$0 = f'(0) = \iiint_\Omega \nabla u \cdot \nabla v dx = -\iiint_\Omega (\Delta u) v dx$$

and hence $\Delta u = 0$ for all $x \in \Omega$.

($\Longleftarrow$): Suppose $u \in A_h$ solves the Dirichlet's problem, let $w \in A_h$, define $v = u - w$, then

$$E(w) = E(u - v) = \frac{1}{2} \iiint_\Omega |\nabla(u - v)|^2 dx = E(u) - \iiint_\Omega \nabla u \cdot \nabla v dx + E(v)$$

use the fact that $v = 0$ on $\partial\Omega$, then

$$0 = \iint_{\partial\Omega} v \cdot \frac{\partial u}{\partial n} dS = \iiint_\Omega \nabla u \cdot \nabla v dx + \iiint_\Omega v \Delta u dx := \iiint_\Omega \nabla u \cdot \nabla v dx$$

which implies $E(w) = E(u) + E(v)$, and $E(v) \geq 0$. ■

Corollary

Corollary 4 (Dirichlet's principle for Poisson's equation)**.** *Define*

$$E(w) := \iiint_\Omega \frac{1}{2} |\nabla w|^2 + w f dx$$

then the last theorem still holds for the Poisson's equation

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = h & \text{on } \partial\Omega. \end{cases}$$

Proof. ($\Longrightarrow$). First suppose u is the minimizer over A_h. Let $v \in C^2(\Omega) \cap C(\bar{\Omega})$ and $v \equiv 0$ on $\partial\Omega$. Then for all $\varepsilon > 0$, $u + \varepsilon v \in A_h$. Define $f(\varepsilon) = E(u + \varepsilon v)$, then we have

$$f'(\varepsilon) = \frac{d}{d\varepsilon} \iiint_\Omega \frac{1}{2} |\nabla(u + \varepsilon v)|^2 + (u + \varepsilon v) f$$

$$= \iiint_\Omega \varepsilon (\nabla v)^2 + \nabla u \nabla v + v f.$$

By definition, u is the minimizer, so f attains a minimum when $\varepsilon = 0$, hence $f'(0) = 0$, which implies

$$\iiint_\Omega \nabla u \nabla v + vf = 0.$$

Finally by Green's identities, we have

$$\begin{aligned}\iiint_\Omega \nabla u \nabla v + vf &= \iint_{\partial\Omega} v\frac{\partial u}{\partial \mathbf{n}} - \iiint_\Omega v\Delta u - vf \\ &= -\iiint v(\Delta u - f) \quad (\text{since } v \equiv 0 \text{ on } \partial\Omega),\end{aligned}$$

hence $\Delta u = f$.

($\Longleftarrow$) Now suppose $u \in A_h$, $\Delta u = f$ in Ω. For all $w \in A_h$, consider $v = u - w$, then $v \in C^2(\Omega) \cap C(\bar{\Omega})$, and $v \equiv 0$ on $\partial\Omega$. Now

$$E(w) = \iiint_\Omega \frac{1}{2}|\nabla(u-v)|^2 + (u-v)f \tag{5.6}$$

$$= E(u) + \iiint_\Omega -\nabla u \nabla v + \frac{1}{2}(\nabla v)^2 - vf \tag{5.7}$$

$$= E(u) - \iint_{\partial\Omega} v\frac{\partial u}{\partial \mathbf{n}} + \iiint_\Omega v\Delta u + \frac{1}{2}(\nabla v)^2 - vf \tag{5.8}$$

$$= E(u) + \iiint_\Omega \frac{1}{2}(\nabla v)^2. \tag{5.9}$$

Hence $E(w) \geq E(u)$. ■

Definition

Definition 14. *The fundamental solution for Laplace equation $\Delta u = 0$ in $\mathbb{R}^n$ is given by the form*

$$\Phi(x) = \begin{cases} \frac{1}{2\pi}\log|x| & n = 2 \\ -\frac{1}{4\pi|x|} & n = 3 \\ -\frac{1}{n(n-2)\omega_n|x|^{n-2}} & n > 3 \end{cases}$$

Theorem

Theorem 14. *Let $n = 3$, then $\Delta\left(\frac{1}{|x|}\right) = -4\pi\delta_0$ in distribution. That is,*

$$-4\pi\phi(0) = \iiint_{\mathbb{R}^3} \frac{1}{|x|}\Delta\phi(x)dx$$

for all $\phi \in C_C^\infty(\mathbb{R}^3)$.

Proof. Let $B(0,\varepsilon)$ be a small ε-ball centered at the origin, then

$$-4\pi\phi(0) = \iiint_{B(0,\varepsilon)} \frac{1}{|x|}\Delta\phi(x)dx + \iiint_{\mathbb{R}^3\setminus B(0,\varepsilon)} \frac{1}{|x|}\Delta\phi(x)dx$$

Since $\phi \in C_C^\infty$, so we may restrict the region to a compactly support set Ω, such that $\Delta\phi(x) \equiv 0$ for all $x \notin \Omega$ (including $\partial\Omega$), i.e

$$-4\pi\phi(0) = \underbrace{\iiint_{B(0,\varepsilon)} \frac{1}{|x|}\Delta\phi(x)dx}_{\text{equation 1}} + \underbrace{\iiint_{\Omega\setminus B(0,\varepsilon)} \frac{1}{|x|}\Delta\phi(x)dx}_{\text{equation 2}}$$

Now for equation 1, assume $|\Delta\phi(x)| \leq K_1$ for all $x \in B(0,\varepsilon)$, we have

$$\begin{aligned}
\iiint_{B(0,\varepsilon)} \frac{1}{|x|}\Delta\phi(x)dx &\leq \iiint_{B(0,\varepsilon)} \frac{1}{|x|} \cdot |\Delta\phi(x)|dx \\
&\leq K_1 \iiint_{B(0,\varepsilon)} \frac{1}{|x|}dx \\
&= K_1 \int_0^\varepsilon \frac{1}{r} \cdot 4\pi r^2 dr \\
&= 2K_1\pi\varepsilon^2.
\end{aligned}$$

hence

$$\lim_{\varepsilon\to 0} \iiint_{B(0,\varepsilon)} \frac{1}{\Delta\phi(x)}dx = 0.$$

Now for equation 2, we apply Green's identity on the two boundaries $\partial B(0,\varepsilon)$ and $\partial\Omega$ and we have

$$\begin{aligned}
\iiint_{\Omega\setminus B(0,\varepsilon)} \left(\frac{1}{|x|}\Delta\phi(x) - \phi(x)\Delta\frac{1}{|x|}\right)dx &= \underbrace{\iint_{\partial\Omega} \left(\frac{1}{|x|}\frac{\partial\phi}{\partial\mathbf{n}} - \phi\frac{\partial\frac{1}{|x|}}{\partial\mathbf{n}}\right)dS}_{\text{equation 3}} \\
&\quad + \underbrace{\iint_{\partial B(0,\varepsilon)} \left(\frac{1}{|x|}\frac{\partial\phi}{\partial\mathbf{n}} - \phi\frac{\partial\frac{1}{|x|}}{\partial\mathbf{n}}\right)dS}_{\text{equation 4}}
\end{aligned}$$

by our choice of Ω, $\phi(x) \equiv 0$ for all $x \in \partial\Omega$ as well as all its derivatives, so equation 3 is identically zero, hence

$$\iiint_{\Omega\setminus B(0,\varepsilon)} \left(\frac{1}{|x|}\Delta\phi(x) - \phi(x)\Delta\frac{1}{|x|}\right)dx = \iint_{\partial B(0,\varepsilon)} \left(\frac{1}{|x|}\frac{\partial\phi}{\partial\mathbf{n}} - \phi\frac{\partial\frac{1}{|x|}}{\partial\mathbf{n}}\right)dS.$$

Note that

$$\left|\frac{\partial\phi}{\partial\mathbf{n}}\right| = |\nabla\Phi\cdot\mathbf{n}| \leq |\nabla\phi|,$$

assume $|\nabla\phi| \leq K_2$ for all $x \in \partial B(0,\varepsilon)$, now we have

$$\begin{aligned}
\iint_{\partial B(0,\varepsilon)} \frac{1}{|x|}\frac{\partial\phi}{\partial\mathbf{n}} &\leq \iint_{\partial B(0,\varepsilon)} \frac{1}{|x|}\left|\frac{\partial\phi}{\partial\mathbf{n}}\right|dS \\
&\leq K_2 \iint_{\partial B(0,\varepsilon)} \frac{1}{|x|}dS \\
&= K_2 \iint_{\partial B(0,\varepsilon)} \frac{1}{\varepsilon}dS \\
&= 4K_2\pi\varepsilon
\end{aligned}$$

hence

$$\lim_{\varepsilon\to 0}\iint_{\partial B(0,\varepsilon)}\frac{1}{|x|}\frac{\partial\phi}{\partial\mathbf{n}}dS=0.$$

Finally we study

$$\begin{aligned}\iint_{\partial B(0,\varepsilon)}\phi\frac{\partial}{\partial\mathbf{n}}\frac{1}{|x|}dS&=\iint_{\partial B(0,\varepsilon)}-\phi\frac{\partial}{\partial r}\frac{1}{r}dS\\&=\iint_{\partial B(0,\varepsilon)}-\frac{\phi(x)}{\varepsilon^2}dS\\&=-4\pi\left(\frac{1}{4\pi\varepsilon^2}\iint_{\partial B(0,\varepsilon)}\phi(x)dS\right).\end{aligned}$$

Eventually we put all terms together:

$$\iiint_{\mathbb{R}^3}\frac{1}{|x|}\Delta\phi(x)dx=\lim_{\varepsilon\to 0}-4\pi\left(\frac{1}{4\pi\varepsilon^2}\iint_{\partial B(0,\varepsilon)}\phi(x)dS\right):=-4\pi\phi(0).$$

■

Note that the convergence result also applies in $\mathbb{R}^2$:

Proposition

Proposition 12. *Let $u(x)=\frac{1}{2\pi}\log|x|$ be the solution to Laplace equation in $\mathbb{R}^2$, then $\Delta(\log|x|)=2\pi\delta_0$ in distribution. That is,*

$$\iint_{\mathbb{R}^2}\log|x|\Delta\phi(x)dx=2\pi\phi(0),$$

for all $\phi(x)\in C_C^\infty(\mathbb{R})$.

Proof. First, note that $\phi(x)\in C_C^\infty(\mathbb{R})$, so we can choose $Supp(\phi(x))=\Omega\subseteq\mathbb{R}^2$, and then

$$\iint_{\mathbb{R}^2}\log|x|\phi(x)dx=\underbrace{\iint_{B(0,\varepsilon)}\log|x|\Delta\phi(x)dx}_{\text{Equation 1}}+\underbrace{\iint_{\Omega\backslash B(0,\varepsilon)}\log|x|\Delta\phi(x)dx}_{\text{Equation 2}}.$$

For equation 1, $\exists K_1\in\mathbb{R}$ such that $|\Delta\phi(x)|\le K_1$ for all $x\in B(0,\varepsilon)$, so

$$\begin{aligned}\iint_{B(0,\varepsilon)}\log|x|\Delta\phi(x)dx&\le K_1\iint_{B(0,\varepsilon)}\log|x|dx\\&=K_1\int_0^\varepsilon 2\pi r\log r dr\quad(dS=2\pi r dr)\\&=K_1\left(\varepsilon^2\log\varepsilon-\frac{1}{2}\varepsilon^2\right).\end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0}\iint_{B(0,\varepsilon)}\log|x|\Delta\phi(x)dx=0,$$

and equation 1 will yield 0 by taking the limit. As for equation 2, we apply Green's identity to both boundaries and we get

$$\iint_{\Omega\setminus B(0,\varepsilon)} \log|x|\Delta\phi(x)dx - \phi(x)\Delta(\log|x|)dx = \underbrace{\iint_{\partial\Omega} \log|x|\frac{\partial\phi(x)}{\partial\mathbf{n}} - \phi(x)\frac{\partial\log|x|}{\partial\mathbf{n}}}_{\text{Equation 3}} + \underbrace{\iint_{\partial B(0,\varepsilon)} \log|x|\frac{\partial\phi(x)}{\partial\mathbf{n}} - \phi(x)\frac{\partial\log|x|}{\partial\mathbf{n}}}_{\text{Equation 4}}.$$

Note that by our choice of Ω, $\phi(x) \equiv 0$ on $\partial\Omega$ so equation 3 wll yield zero, hence we only have equation 4 left. For the first term in equation 4, note that

$$\left|\frac{\partial\phi(x)}{\partial n}\right| \leq |\nabla\phi(x)\cdot\mathbf{n}| \leq K_2,$$

using the fact that $\phi(x) \in C_C^\infty(\mathbb{R})$. Hence

$$\iint_{\partial B(0,\varepsilon)} \log|x|\frac{\partial\phi(x)}{\partial\mathbf{n}}dx \leq K_2 \iint_{\partial B(0,\varepsilon)} \log|x|dx = K_2 2\pi\varepsilon\log\varepsilon,$$

by taking the limit $\varepsilon \to 0$, this term will also yield zero. The final term can be computed by

$$\begin{aligned}
\iint_{\partial B(0,\varepsilon)} -\phi(x)\frac{\partial\log|x|}{\partial\mathbf{n}}dx &= \iint_{\partial B(0,\varepsilon)} \phi(r)\cdot\frac{\partial}{\partial r}\log r dr \\
&= \iint_{\partial B(0,\varepsilon)} \phi(r)\frac{1}{\varepsilon}dr \\
&= 2\pi\left(\frac{1}{2\pi\varepsilon}\iint_{\partial B(0,\varepsilon)} \phi(r)dr\right) \\
&= 2\pi\phi(0) \quad \text{by averaging lemma.}
\end{aligned}$$

Thus putting all term together and take the limit $\varepsilon \to 0$, we conclude that

$$\iint_{\mathbb{R}^2} \log|x|\Delta\phi(x)dx = 2\pi\phi(0).$$

■

5.2 Green's Identities

Some remarks on calculus:

Theorem

Theorem 15 (Divergence Theorem)**.** *Let* **F** *be a smooth vector field on a bounded domain* Ω *with outer normal* **n***, then*

$$\iiint_\Omega \nabla\cdot\mathbf{F}dV = \iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}dS$$

Pointwise Divergence Theorem: Given a smooth function $f(x)$ on a bounded domain $\Omega \subseteq \mathbb{R}^3$, then

$$\iiint_\Omega f_{x_i}(x)dx = \iint_{\partial\Omega} f n_i dS$$

where n_i is the i-th component of the outer unit normal $\mathbf{n}$.

Theorem

Theorem 16 (Integration by Parts)**.**

$$\iint_\Omega \mathbf{u}\cdot\nabla v dx = -\iiint (\nabla\cdot\mathbf{u})v dx + \iint_{\partial\Omega} (v\mathbf{u})\cdot\mathbf{n} dS$$

Theorem

Theorem 17 (Green's Identities)**.**

$$\iiint_\Omega v\Delta u = \iint_{\partial\Omega} v\frac{\partial u}{\partial\mathbf{n}} - \iiint_\Omega \nabla u\cdot\nabla v$$

$$\iiint_\Omega (v\Delta u - u\Delta v) = \iint_{\partial\Omega} v\frac{\partial u}{\partial\mathbf{n}} - u\frac{\partial v}{\partial\mathbf{n}}$$

In $\mathbb{R}^3$, we can derive the representation formula from Green's identity. The idea is that any harmonic function can be written as an intergal over the boundary.

Theorem

Theorem 18 (Representation Formula)**.** *Let u be harmonic in $D \subseteq \mathbb{R}^3$, then for all $\mathbf{x}_0 \in D$,*

$$u(\mathbf{x}_0) = \iint_{\partial D}\left[-u(x)\frac{\partial}{\partial\mathbf{n}}\frac{1}{|\mathbf{x}-\mathbf{x}_0|} + \frac{1}{|\mathbf{x}-\mathbf{x}_0|}\frac{\partial u}{\partial\mathbf{n}}\right]\frac{dS}{4\pi}.$$

Proof. Let $v(x) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}_0|}$, then we know $v(\mathbf{x})$ is the fundamental solution of the harmonic equation in $\mathbb{R}^3$ hence $\Delta v = 0$, combine with $\Delta u = 0$ we use Green's second identity to get

$$\iint_{\partial D}\left[-u(\mathbf{x})\frac{\partial}{\partial\mathbf{n}}\frac{1}{|\mathbf{x}-\mathbf{x}_0|} + \frac{1}{|\mathbf{x}-\mathbf{x}_0|}\frac{\partial u}{\partial\mathbf{n}}\right]\frac{dS}{4\pi} = 0.$$

For simplicity, let $\mathbf{x}_0 = \mathbf{0}$ be the origin, again we consider a ε-ball centered around $\mathbf{x}_0$, and let $r = |\mathbf{x}|$, $v(\mathbf{x}) = -1/(4\pi r)$, then have

$$\iint_{\partial D}\left[u\frac{\partial}{\partial\mathbf{n}}\frac{1}{r} - \frac{\partial u}{\partial\mathbf{n}}\frac{1}{r}\right]dS = \iint_{\partial B(x_0,\varepsilon)}\left[u\frac{\partial}{\partial r}\frac{1}{r} - \frac{\partial u}{\partial r}\frac{1}{r}\right]dS,$$

since $\partial/\partial n = -\partial/\partial r$ on the boundary of the ε-ball. Then it can be computed using the same idea as the previous theorems. ■

Proposition

Proposition 13 (Representation Formula in 2D). *Suppose u is harmonic in $D \subseteq \mathbb{R}^2$, then for all $\mathbf{x}_0 \in D$,*

$$u(\mathbf{x}_0) = \frac{1}{2\pi}\int_{\partial D}\left[u(\mathbf{x})\frac{\partial}{\partial n}\log|\mathbf{x}-\mathbf{x}_0| - \frac{\partial u}{\partial n}\log|\mathbf{x}-\mathbf{x}_0|\right]dS$$

5.3 Green's Functions

Definition

Definition 15. *The Green's function $G(x)$ for the Laplacian and $x_0 \in D$ is a function defined for $x \in D$ such that:*

(i) $G(x)$ possesses continuous second derivatives and $\Delta G = 0$ in D except at $x = x_0$.

(ii) $G(x) = 0$ for $x \in \partial D$.

(iii) The function $G(x) + \frac{1}{4\pi}\frac{1}{|x-x_0|}$ is finite at x_0 and has continuous second derivatives everywhere and is harmonic at x_0.

We will use the usual notation for Green's function as $G(x,x_0)$.

Theorem

Theorem 19. *If $G(x,x_0)$ is the Green's function, then the solution of the Dirichlet problem is given by the formula*

$$u(x_0) = \iint_{\partial D} u(x)\frac{\partial G(x,x_0)}{\partial \mathbf{n}}dS.$$

Proof. Let $v(\mathbf{x}) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}_0|}$, by the construction of $G(\mathbf{x},\mathbf{x}_0)$, the function $H(\mathbf{x}) = -v(\mathbf{x}) + G(\mathbf{x},\mathbf{x}_0)$ is harmonic. Using Green's second identity, we have

$$0 \equiv \iiint_D u(x)\Delta H(\mathbf{x}) - H(\mathbf{x})\Delta u(\mathbf{x}) = \iint_{\partial D} u(\mathbf{x})\frac{\partial H(\mathbf{x})}{\partial \mathbf{n}} - H(\mathbf{x})\frac{\partial u(\mathbf{x})}{\partial \mathbf{n}}dS, \tag{5.10}$$

which is because $\Delta H = \Delta u = 0$. Using representation formula, we also have

$$u(\mathbf{x}_0) = \iint_{\partial D} u(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}} - v(\mathbf{x})\frac{\partial u(\mathbf{x})}{\partial \mathbf{n}}dS. \tag{5.11}$$

Then combine with (5.10), (5.11) and the fact that $G(\mathbf{x}) = v(\mathbf{x}) + H(\mathbf{x})$, we have

$$u(\mathbf{x}_0) = \iint_{\partial D} u(\mathbf{x})\frac{\partial G(\mathbf{x},\mathbf{x}_0)}{\partial \mathbf{n}} - G(\mathbf{x},\mathbf{x}_0)\frac{\partial \mathbf{u}(\mathbf{x})}{\partial \mathbf{n}}dS.$$

By definition, $G(\mathbf{x},\mathbf{x}_0) \equiv 0$ on ∂D so we arrive at

$$u(x_0) = \iint_{\partial D} u(x)\frac{\partial G(x,x_0)}{\partial \mathbf{n}}dS.$$

■

Theorem

Theorem 20. *For any region D, the Green's function is always symmetric, i.e.,*

$$G(x,x_0) = G(x_0,x), \quad x \neq x_0.$$

Proof. ■

Theorem

Theorem 21 (Solution of Poisson's Equation)**.** *The solution to the Poisson's equation $\Delta u = f$ in D and $u = h$ on ∂D is given by*

$$u(x_0) = \iint_{\partial D} h(x) \frac{\partial G(x,x_0)}{\partial \mathbf{n}} dS + \iiint_D f(x) G(x,x_0) dx.$$

Proof. ■

5.4 Green's Function for Halfspace and Sphere

Finding Green's function is not easy. In this section we will derive the Green's function for two easy cases and hence solve the harmonic functions with those boundary conditions.

5.4.1 Halfspace

A halfspace in $\mathbb{R}^3$ can be viewed as the region

$$D = \{(x,y,z) : z > 0\}.$$

So for each $\mathbf{x} = (x,y,z) \in D$, we can find its *reflected point* $\mathbf{x}^* = (x,y,-z) \notin D$. Recall that the fundamental solution

$$u(\mathbf{x}) = -\frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}_0|}$$

is harmoinc in D, and it already satisfies (i), (iii) from the definition of a Green's function. We now propose a new function based on that to make it satisfy (ii) as well.

Proposition

Proposition 14. *Define*

$$G(\mathbf{x},\mathbf{x}_0) = -\frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}_0|} + \frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}_0^*|}.$$

Then G is the Green's function on $D := \{(x,y,z) : z > 0\}$.

Proof. We will verify all three conditions of a Green's function.

(i) Clearly G is finite and differentiable except at $\mathbf{x}_0$, and $\Delta G = 0$ since it is a sum of fundamental solutions.

(ii) Let $\mathbf{x} \in \partial D$, i.e., $z = 0$, then it is clear that $|\mathbf{x} - \mathbf{x}_0| = |\mathbf{x} - \mathbf{x}_0^*|$, hence $G(\mathbf{x}, \mathbf{x}_0) = 0$.

(iii) Since $\mathbf{x}_0^* \notin D$, so the function $\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}_0^*|}$ has no singularity in D, and hence G has proper singularity at $\mathbf{x}_0$. ■

Equipped with G, we now solve the Dirichlet problem for halfspace:

$$\Delta u = 0, z > 0 \quad u(x,y,0) = h(x,y)$$

On the boundary $z = 0$, we have $\partial G/\partial \mathbf{n} = -\partial G/\partial z|_{z=0}$ (we need outward pointing normal and D is the positive part of the halfplane), and we have

$$-\frac{\partial G}{\partial z} = \frac{1}{4\pi}\left(\frac{z+z_0}{|\mathbf{x}-\mathbf{x}_0^*|^3} - \frac{z-z_0}{|\mathbf{x}-\mathbf{x}_0^*|^3}\right) = \frac{1}{2\pi}\frac{z_0}{|\mathbf{x}-\mathbf{x}_0|^3} \quad \text{on } z = 0.$$

Hence the solution is given by

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi}\iint [(x-x_0)^2 + (y-y_0)^2 + z_0^2]^{-3/2} h(x,y)\,dxdy$$

which is given by

$$u(\mathbf{x}_0) = \frac{z_0}{2\pi}\iint_{\partial D} \frac{h(\mathbf{x})}{|\mathbf{x}-\mathbf{x}_0|^3} dS.$$

Theorem

Theorem 22. *The formula that solves the Dirichlet problem for the half-space is given by*

$$u(\mathbf{x}_0) = \frac{z_0}{2\pi}\iint_{\partial D} \frac{h(\mathbf{x})}{|\mathbf{x}-\mathbf{x}_0|^3} dS.$$

5.4.2 Sphere

We define the region D as a sphere:

$$D := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| < a\}.$$

For any non-zero point $\mathbf{x}_0 \in D$, we may define its reflection point in a similiar way:

$$\mathbf{x}_0^* = \frac{a^2 \mathbf{x}_0}{|\mathbf{x}_0|^2}.$$

Definition

Definition 16. *The Green's function for the sphere D is given by*

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}_0|} + \frac{a}{4\pi|\mathbf{x}_0|}\frac{1}{|\mathbf{x}-\mathbf{x}_0^*|}.$$

In the case $\mathbf{x}_0 = \mathbf{0}$, we may simplify the formula as

$$G(\mathbf{x},\mathbf{0}) = -\frac{1}{4\pi|\mathbf{x}|} + \frac{1}{4\pi a}.$$

And now we solve the Dirichlet poblem

$$\Delta u = 0, |\mathbf{x}| < a \quad u = h, |\mathbf{x}| = a$$

Theorem

Theorem 23. *The formula that solves the Dirichlet problem for the sphere is given by*

$$u(\mathbf{x}_0) = \frac{a^2 - |\mathbf{x}_0|^2}{4\pi a} \iint_{\partial D} \frac{h(\mathbf{x})}{|\mathbf{x}-\mathbf{x}_0|^3} dS.$$

We can use spherical coordinate to represent the formula above as

$$u(r_0,\theta_0,\phi_0) = \frac{a(a^2-r_0^2)}{4\pi} \int_0^{2\pi} \int_0^{\pi} \frac{h(\theta,\phi)}{(a^2+r_0^2-2ar_0\cos\phi)^{3/2}} \sin\theta d\theta d\phi.$$

In $2D$, we may also derive the Green's function for a Disc: $D := \{(x,y), x^2+y^2 < a^2\}$. Consider $\mathbf{x}_0 \in D$ and we define its reflected point by $\mathbf{x}_0^* := \frac{a^2}{|\mathbf{x}_0|^2}\mathbf{x}_0 \notin D$. Let

$$G(\mathbf{x},\mathbf{x}_0) = \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}_0| - \frac{1}{2\pi}\log\left(\frac{|\mathbf{x}_0|}{a}|\mathbf{x}-\mathbf{x}_0^*|\right), \tag{5.12}$$

where we already can verify condition (i), (iii) from Green's function. We suffices to show $G(\mathbf{x},\mathbf{x}_0) = 0$ on ∂D. Choose $\mathbf{x} \in \partial D, |\mathbf{x}| = a$, then we may use a congruent triangle statement to show that

$$|\mathbf{x}-\mathbf{x}_0| = \left|\frac{|\mathbf{x}_0|}{a}\mathbf{x} - \frac{a}{|\mathbf{x}_0|}\mathbf{x}_0\right| \tag{5.13}$$

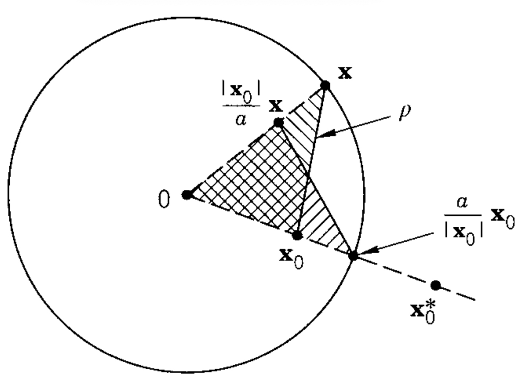

Figure 5: The congruent triangles formed by vertices $\mathbf{0},\mathbf{x},\mathbf{x}_0$ and $\mathbf{0}, \frac{a}{|\mathbf{x}_0|}\mathbf{x}_0, \frac{|\mathbf{x}_0|}{a}\mathbf{x}$. (*Credit: Strauss PDE, Chapter 7.4, Page 194.*)

Hence in (5.13), we have

$$|\mathbf{x}-\mathbf{x}_0| = \frac{|\mathbf{x}_0|}{a}\left|\mathbf{x} - \frac{a^2}{|\mathbf{x}_0|^2}\mathbf{x}_0\right| = \frac{|\mathbf{x}_0|}{a}|\mathbf{x}-\mathbf{x}_0^*|. \tag{5.14}$$

So use the relation in (5.14) and plug into $G(\mathbf{x},\mathbf{x}_0)$ defined in (5.12), we see that $G(\mathbf{x},\mathbf{x}_0)=0$.

Note that the idea is exatly the same when we derive the Green's function for a ball where

$$G(\mathbf{x},\mathbf{x}_0) = -\frac{1}{4\pi}|\mathbf{x}-\mathbf{x}_0| + \frac{1}{4\pi}\frac{a}{|\mathbf{x}_0|}|\mathbf{x}-\mathbf{x}_0^*|. \tag{5.15}$$

The difference in the second term is because the difference in fundamental solution. Now let us prove theorem 23 as well as derive the fundamental solution in a 2D case.

First consider the Dirichlet problem in a ball

$$\Delta u = 0, |\mathbf{x}| < a \quad u = h, |\mathbf{x}| = a.$$

Let $\rho^2 = |\mathbf{x}-\mathbf{x}_0|^2$, then $2(\mathbf{x}-\mathbf{x}_0) = 2\rho\nabla\rho$, similarly we have $\nabla\rho^* = (\mathbf{x}-\mathbf{x}_0^*)/\rho^*$ where $\rho^* = |\mathbf{x}-\mathbf{x}_0^*|$. So differentiate (5.15) yields

$$\begin{aligned}\nabla G(\mathbf{x},\mathbf{x}_0) &= \frac{1}{4\pi}\frac{\mathbf{x}-\mathbf{x}_0}{\rho^3} - \frac{1}{4\pi}\frac{a}{|\mathbf{x}_0|}\frac{\mathbf{x}-\mathbf{x}_0^*}{\rho^{*3}} \\ &= \frac{1}{4\pi\rho^3}\left[\mathbf{x}-\mathbf{x}_0-\left(\frac{|\mathbf{x}_0|}{a}\right)^2\mathbf{x}+\mathbf{x}_0\right].\end{aligned}$$

Hence on ∂D we have

$$\frac{\partial G}{\partial \mathbf{n}} = \frac{\mathbf{x}}{a}\cdot\nabla G(\mathbf{x},\mathbf{x}_0) = \frac{a^2-|\mathbf{x}_0|^2}{4\pi a\rho^3},$$

which yields

$$u(\mathbf{x}_0) = \frac{a^2-|\mathbf{x}_0|^2}{4\pi a}\iint_{\partial D}\frac{h(\mathbf{x})}{|\mathbf{x}-\mathbf{x}_0|^3}dS.$$

So we may apply the same idea for the Dirichlet problem in a disc:

$$\Delta u = 0, |\mathbf{x}| < a \quad u = h, |\mathbf{x}| = a,$$

where the Green's function for a disc is given by

$$G(\mathbf{x},\mathbf{x}_0) = \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}_0| - \frac{1}{2\pi}\log\left(\frac{|\mathbf{x}_0|}{a}|\mathbf{x}-\mathbf{x}_0^*|\right).$$

We differentiate the equation above, and use $\rho = |\mathbf{x}-\mathbf{x}_0|, \rho^* = |\mathbf{x}-\mathbf{x}_0^*|$, we have

$$\begin{aligned}\nabla G(\mathbf{x},\mathbf{x}_0) &= \frac{1}{2\pi}\frac{1}{\rho}\frac{\mathbf{x}-\mathbf{x}_0}{\rho} - \frac{1}{2\pi}\frac{1}{\rho^*}\frac{\mathbf{x}-\mathbf{x}_0^*}{\rho^*} \\ &= \frac{1}{2\pi\rho^2}\left[\mathbf{x}-\mathbf{x}_0-\left(\frac{|\mathbf{x}_0|}{a}\right)^2\mathbf{x}+\mathbf{x}_0\right].\end{aligned}$$

Hence on ∂D where $|\mathbf{x}|=1$, we have the out-pointing unit normal vector given by

$$\frac{\partial G}{\partial \mathbf{n}} = \frac{\mathbf{x}}{a}\cdot\nabla G(\mathbf{x},\mathbf{x}_0) = \frac{a^2-|\mathbf{x}_0|^2}{2\pi a\rho^2}$$

then it follows that

$$u(\mathbf{x}_0) = \frac{a^2 - |\mathbf{x}_0|^2}{2\pi a} \int_{\partial D} \frac{h(\mathbf{x})}{|\mathbf{x} - \mathbf{x}_0|^2} ds.$$

Theorem

Theorem 24 (Poisson's Problem)**.** *The solution of the Poisson's problem*

$$\Delta u = f, \mathbf{x} \in D \quad u = h, \mathbf{x} \in \partial D$$

is given by

$$u(\mathbf{x}_0) = \iint_{\partial D} h(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial \mathbf{n}} dS + \iiint_D f(\mathbf{x}) G(\mathbf{x}, \mathbf{x}_0) d\mathbf{x}.$$

Proof. First let G be the corresponding Green's function, we have

$$\Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0), \mathbf{x} \in D, \quad G(\mathbf{x}, \mathbf{x}_0) = 0, \mathbf{x} \in \partial D.$$

Then using Green's second identity we have that

$$\iiint_D G\Delta u - u\Delta G = \iint_{\partial D} G\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial G}{\partial \mathbf{n}} dS,$$

use boundary conditions and properties of G we have that

$$\iiint_D G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}) d\mathbf{x} - \iiint_D u(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}_0) d\mathbf{x} = -\iint_{\partial D} h(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial \mathbf{n}} dS = 0,$$

hence

$$u(\mathbf{x}_0) = \iint_{\partial D} h(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial \mathbf{n}} dS + \iiint_D f(\mathbf{x}) G(\mathbf{x}, \mathbf{x}_0) d\mathbf{x}.$$

∎

Boundary Problems

6.1 Separation of Variables and the Dirichlet Condition

We first consider the homogeneous Dirichlet conditions for the wave equation:

$$u_{tt} = c^2 u_{xx} \quad (0 < x < l), \quad u(0,t) = 0 = u(l,t)$$

with initial conditions $u(x,0) = \phi(x), u_t(x,0) = \psi(x)$. The method we shall use consists of building up the general solution as a linear combination of special ones that are easy to find. We say a *separated solution* is a solution of the wave equation defined above taking the form

$$u(x,t) = X(x)T(t)$$

by assuming a solution of the form (6.2), we have

$$X(x)T''(t) = c^2 X''(x)T(t) \implies -\frac{T''}{c^2 T} = -\frac{X''}{X} = \lambda$$

where λ is a constant since $\frac{\partial \lambda}{\partial x} = \frac{\partial \lambda}{\partial t} = 0$, we will later show that $\lambda > 0$, so define $\lambda = \beta^2, \beta > 0$, so we have a pair of separable ODEs:

$$X'' + \beta^2 X = 0 \quad T'' + c^2\beta^2 T = 0$$

which leads to the solution

$$X(x) = C\cos\beta x + D\sin\beta x \quad T(t) = A\cos\beta ct + B\sin\beta ct$$

we now want to impose boundary condition that $u(0,t) = u(l,t) = 0$, i.e $X(0) = X(l) = 0$, so $X(0) = C, X(l) = D\sin\beta l$, and $\beta l = n\pi$, so for each n we have a separate solution for X, denote by $X_n(x)$, and so $u_n(x,t) = X_n(x)T_n(t)$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, X_n(x) = \sin\frac{n\pi x}{l}$$

and

$$u_n(x,t) = \left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

so we can write the general solution as a sum of u_n:

$$u(x,t) = \sum_n \left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}.$$

This formula will solve the wave equation with boundary condition

$$u_{tt} = c^2 u_{xx}, u(0,t) = u(l,t) = 0, u(x,0) = \phi(x), u_t(x,0) = \psi(x)$$

if

$$\phi(x) = \sum_n A_n\sin\frac{n\pi x}{l} \quad \psi(x) = \sum_n \frac{n\pi c}{l} B_n\sin\frac{n\pi x}{l}$$

Later, we will see Fourier series, and then (6.10) will become more clear.

We may also consider the diffusion equation

$$\textbf{DE:} u_t = ku_{xx}(0 < x < l, 0 < t < \infty), \quad \textbf{BC:} u(0,t) = u(l,t) = 0, \quad \textbf{IC:} u(x,0) = \phi(x)$$

we may separate variables as before, let $u(x,t) = X(x)T(t)$ and get the following

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

So we have $T' = -\lambda kT$ and the solution is given by $T(t) = Ae^{-\lambda kt}$, also $-X'' = \lambda X$ is the exact same equation we get as in the wave equation, so in general we have

$$u(x,t) = \sum_n A_n e^{-(n\pi/l)^2 kt} \sin\frac{n\pi x}{l}$$

and this solves (6.11) if

$$\phi(x) = \sum_n A_n \sin\frac{n\pi x}{l}.$$

The numbers $\lambda_n = (n\pi/l)^2$ are called the *eigenvalues*, the functions $X_n(x) = \sin(n\pi x/l)$ are called the *eogenfunctions*. The reason is that they satisfy

$$-\frac{d^2}{dx^2}X = \lambda X, \quad X(0) = X(l) = 0$$

Example

Example 17. *Consider a metal rod of length l, insulated along its sides except the ends. The rod has tempreture $T = 1$ and suddenly both ends are plunged into a bath of $T = 0$. In this seneario write down the differential equations and write out the general solution.*

Solution 17. *The rod satisfies heat equation, where we have*

$$u_t = ku_{xx}, u(0,t) = u(l,t) = 0, u(x,0) = 1.$$

By separation of variables, we seek a solution of the form $u(x,t) = X(x)T(t)$, so

$$u_t = X(x)T'(t) = ku_{xx} = kX''(x)T(t) \implies -\frac{T'(t)}{kT(t)} = -\frac{X''(x)}{X(x)}$$

which means the ratio must be a constant since X, T are independent. So we will have two separate ODEs:

$$T'(t) = -k\lambda T(t) \quad X''(x) = -\lambda X(x) \quad \lambda \in \mathbb{R}.$$

Here λ is called the eigenvalue. Later we will see that, under a symmetric boundary condition and theorem 19,20, all eigenvalues must be non-negative. Which means $\lambda \geq 0$. If $\lambda = 0$ then we will only have the trivial solution. So assume $\lambda > 0$, and the above two ODEs can be easily solved as follows:

$$T'(t) = -k\lambda T(t) \implies T(t) = Ce^{-\lambda kt}, \quad C \in \mathbb{R};$$

$$X''(x) + \lambda X(x) = 0 \implies X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x).$$

We now equate the boundary conditions:

$$u(0,t) = 0 \implies X(0) = 0 \implies B \equiv 0.$$

$$u(l,t) = 0 \implies X(l) = 0 \implies A\sin(\sqrt{\lambda}x) = 0 \implies \sqrt{\lambda} = \frac{n\pi}{l}, n \in \mathbb{N}.$$

$$u(x,0) = 1 \implies T(0) = 1 \overset{\text{plug in values for } \lambda, C}{\implies} T(t) = e^{-(n\pi/l)^2 kt}.$$

We hence have the following set of eigenvalues and eigenfunctions:

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \quad X_n = \sin\frac{n\pi x}{l}, \quad n \in \mathbb{N}.$$

The general solution can then be viewed as

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin\frac{n\pi x}{l}.$$

Since $u(x,t) = X(x)T(t)$, so now our goal is to find A_n where

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \equiv \phi(x) = 1.$$

Those A_ns are exactly the coefficients of Fourier sine series of $\phi(x) = 1$, and it is easy to see that

$$1 = \sum_{n=1}^{\infty} \frac{4}{\pi}\left(\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{l}\right).$$

So we have

$$A_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd} \end{cases}.$$

Example

Example 18. *Suppose we have a partical on $(0,l)$ satisfying the Schrödinger's equation $u_t = iu_{xx}$ ($i^2 = -1$) with Dirichlet boundary condition (that is, $u(0,t) = u(l,t) = 0$). Find a series solution to this problem.*

Solution 18. *Again, we perform separation of variables:*

$$u(x,t) = X(x)T(t),$$

and

$$u_t = X(x)T'(t) = iu_{xx} = iX''(x)T(t) \implies -\frac{T'(t)}{iT(t)} = -\frac{X''(x)}{X(x)} = \lambda$$

as in the previous example. With symmetric boundary condition and by theorem 19,20 (see later), the eigenvalues are non-negative, and here we may assume eigenvalues are positive otherwise we have only the trivial solution. Note that the solution for X is the same as the previous example, so we have

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \quad X_n(x) = \sin\frac{n\pi x}{l}, \quad n \in \mathbb{N}.$$

As for T, we simply have

$$T'(t) + i\lambda T(t) = 0 \implies T(t) = Ce^{-i\lambda t}$$

equip with boundary condition, plug in λ, we have

$$T(t) = e^{-(n\pi/l)^2 it}.$$

So the general solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 it} \sin\frac{n\pi x}{l}.$$

Example

Example 19. *Consider waves in a resistant medium satisfying the following equation on $(0,l)$:*

$$u_{tt} = c^2 u_{xx} - ru_t, u(0,t) = u(l,t) = 0, u(x,0) = \phi(x), u_t(x,0) = \psi(x). \quad (r \in \mathbb{R})$$

Find a series solution to the problem for:

(i) $0 < r < 2\pi c/l$;

(ii) $2\pi c/l < r < 4\pi c/l$.

Solution 19. *Using separation of variables, $u(x,t) = X(x)T(t)$ hence $u_{tt} = c^2 u_{xx} - ru_t$ can be written as*

$$X(x)T''(t) = c^2 X''(x)T(t) - rX(x)T'(t) \implies -\frac{T''(t)}{c^2T(t)} - \frac{rT'(t)}{c^2T(t)} = -\frac{X''(x)}{X(x)} = \lambda, \lambda \in \mathbb{R}.$$

The solution for $X(x)$ is simply given by

$$X(x) = A\sin\sqrt{\lambda}t + B\cos\sqrt{\lambda}t,$$

we claim that all eigenvalues are non-negative (theorem 20), and in this case it suffices to consider positive eigenvalues hence equip with boundary condition we get $B \equiv 0$ and the following eigenvalues and eigenfunctions:

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \quad X_n = \sin\frac{n\pi x}{l}.$$

As for T, we have the ODE

$$T''(t) + rT'(t) + \lambda c^2 T(t) = 0$$

whose auxiliary equation is given by

$$x^2 + rx + \lambda c^2 = 0 \implies \Delta = r^2 - 4\lambda c^2$$

- *Under (i), we have*

$$\Delta \le 4c^2\left(\frac{\pi^2}{l^2} - \lambda\right) < 0, \quad \textit{since } \lambda_n = \frac{n^2\pi^2}{l^2}, n \ge 1.$$

So the solutions are given by

$$x = -\frac{r}{2} \pm \frac{\sqrt{4\lambda c^2 - r^2}}{2} i.$$

So the solution to T is simply

$$T(t) = Ce^{-\frac{r}{2}t} \sin\left(\frac{\sqrt{4\lambda c^2 - r^2}}{2} t\right) + De^{-\frac{r}{2}t} \cos\left(\frac{\sqrt{4\lambda c^2 - r^2}}{2} t\right).$$

Now, for each λ_n, we have

$$T_n(t) = e^{-\frac{r}{2}t} \left[C_n \sin\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} t\right) + D_n \cos\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} t\right)\right].$$

Hence the general solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) \tag{6.1}$$

$$= \sum_{n=1}^{\infty} e^{-\frac{r}{2}t} \left[C_n \sin\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} t\right) + D_n \cos\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} t\right)\right] \sin\frac{n\pi x}{l}. \tag{6.2}$$

- *The case in (ii) can be done in a similar way.*

Fourier Series

7.1 The Coefficients for Sine and Cosine Series

Definition

Definition 17. *On the interval* $(0,l)$, *we define the Fourier sine series as*

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$$

Later we will show that for any function defined $\phi(x)$ defined on $(0,l)$ then the Fourier since series converges to $\phi(x)$. Our task is try to find the coefficients A_n first.

Lemma 2. *If* $m \neq n$, *then*

$$\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = 0$$

Proof. We use the trig-identity that

$$\sin(x+y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y).$$

■

Now fix m, we have

$$\int_0^l \phi(x)\sin\frac{m\pi x}{l}dx = \int_0^l \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx$$

by interchaning sum and integral (we may do that), and use the fact that when $m \neq n$ the integral is indeed zero, we have

$$\int_0^l \phi(x)\sin\frac{m\pi x}{l}dx = A_m \int_0^l \sin^2\frac{m\pi x}{l}dx$$

and by direct computation we have

$$\int_0^l \sin^2\frac{m\pi x}{l}dx = \int_0^l \frac{1}{2} - \frac{1}{2}\cos\frac{2m\pi x}{l}dx \tag{7.1}$$

$$= \left(\frac{1}{2}x - \frac{l}{4m\pi}\sin\frac{2m\pi x}{l}\right)\Bigg|_{x=0}^{l} \tag{7.2}$$

$$= \frac{1}{2}l. \tag{7.3}$$

Hence we have the coefficients of Fourier sine series given by

$$\boxed{A_m = \frac{2}{l}\int_0^l \phi(x)\sin\frac{m\pi x}{l}dx.}$$

Recall the wave equation with Dirichlet conditions:

$$u_{tt} = c^2 u_{xx}, (0 < x < l), \quad u(0,t) = u(l,t) = 0, \quad u(x,0) = \phi(x), y_t(x,0) = \psi(x),$$

if

$$\phi(x) = \sum_n A_n \sin\frac{n\pi x}{l} \qquad \psi(x) = \sum_n \frac{n\pi c}{l} B_n \sin\frac{n\pi x}{l}$$

then the solution takes the form

$$u(x,t) = \sum_n \left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}.$$

where now we can use the formula derived to find the coefficients of A_n, B_n:

$$A_n = \frac{2}{l}\int_0^l \phi(x)\sin\frac{m\pi x}{l}dx \qquad \frac{m\pi c}{l}B_m = \frac{2}{l}\int_0^l \psi(x)\sin\frac{m\pi x}{l}dx.$$

Definition

Definition 18. *We now define the Fourier cosine series on the interval* $(0,l)$ *by*

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$$

Similarly, if $m \neq n$ we also have the property that

$$\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}dx = 0$$

and we can see that

$$\int_0^l \phi(x)\cos\frac{m\pi x}{l}dx = A_m\int_0^l \cos^2\frac{m\pi x}{l}dx = \begin{cases} \frac{1}{2}lA_m & m \neq 0 \\ \frac{1}{2}lA_0 & m = 0 \end{cases}$$

Hence the coefficients for the cosine series are given by

$$\boxed{A_m = \frac{2}{l}\int_0^l \phi(x)\cos\frac{m\pi x}{l}dx}.$$

Definition

Definition 19. *Now, the full Fourier series of* $\phi(x)$ *on the interval* $-l < x < l$ *is defined as*

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{l} + B_n\sin\frac{n\pi x}{l}\right).$$

where

$$\boxed{A_n = \frac{1}{l}\int_{-l}^{l}\phi(x)\cos\frac{n\pi x}{l}dx}$$

$$\boxed{B_n = \frac{1}{l}\int_{-l}^{l}\phi(x)\sin\frac{n\pi x}{l}dx}$$

Example

Example 20. *Let $\phi(x) = 1$ in the interval $(0,l)$, we now find its Fourier series:*

Solution 20. *First the Fourier sine series of $\phi(x) = 1$ is given by*

$$1 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

where using the formula for the coefficients, we have

$$A_n = \frac{2}{l}\int_0^l 1 \cdot \sin \frac{n\pi x}{l} dx = -\frac{2}{n\pi} \cos \frac{n\pi x}{l}\bigg|_{x=0}^{x=l} = \begin{cases} 0 & n \text{ is even} \\ \frac{4}{n\pi} & n \text{ is odd} \end{cases}.$$

Thus we ca express $\phi(x) = 1$ on $(0,l)$ as

$$1 = \frac{4}{\pi}\left(\sin \frac{\pi x}{l} + \frac{1}{3}\sin \frac{3\pi x}{l} + \frac{1}{5}\sin \frac{5\pi x}{l} + \cdots\right).$$

we may also compute the coefficients for the Fourier cosine series:

$$A_n = \frac{2}{l}\int_0^l \cos \frac{n\pi x}{l} dx = \frac{2}{m\pi} \sin \frac{m\pi x}{l}\bigg|_{x=0}^{x=l} \equiv 0, m \neq 0$$

hence the Fourier cosine series is trivial:

$$1 = 1$$

Example

Example 21. *Let $\phi(x) = x$ in the interval $(0,l)$. We now find its Fourier series:*

Solution 21. *First the Fourier sine series is given by*

$$x = \sum_{n=1}^{\infty} A_n \sin \frac{\pi n x}{l}$$

where the coefficients are given by

$$A_n = \frac{2}{l}\int_0^l x \sin \frac{\pi n x}{l} dx$$

note that using integration by parts we have

$$\int_0^l x \sin \frac{\pi n x}{l} dx = -\frac{l}{\pi n} x \cos \frac{\pi n x}{l}\bigg|_0^l + \int_0^l \frac{l}{\pi n}\cos \frac{\pi n x}{l} dx \tag{7.4}$$

$$= (-1)^{n+1}\frac{l^2}{\pi n} + \frac{l^2}{\pi^2 n^2}\sin \frac{\pi n x}{l}\bigg|_{x=0}^{x=l} \tag{7.5}$$

$$= (-1)^{n+1}\frac{l^2}{\pi n}. \tag{7.6}$$

Hence we have

$$A_n = (-1)^{n+1}\frac{2l}{\pi n}$$

and we may express $\phi(x) = x$ as its Fourier sine series:

$$x = \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \cdots\right).$$

we now express $\phi(x) = x$ as a Fourier cosine series:

$$x = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{\pi n x}{l}$$

and the formula gives us

$$A_0 = \frac{2}{l}\int_0^l x dx = l \tag{7.7}$$

and when $n \geq 1$ we have

$$\begin{aligned}
A_n &= \frac{2}{l}\int_0^l x\cos\frac{\pi n x}{l}dx && (7.8)\\
&= \frac{2}{l}\left[\frac{l}{\pi n}x\sin\frac{\pi n x}{l}\bigg|_{x=0}^{x=l} - \int_0^l \frac{l}{\pi n}\sin\frac{\pi n x}{l}dx\right] && (7.9)\\
&= \frac{2}{l}\left[\frac{l^2}{\pi^2 n^2}\cos\frac{\pi n x}{l}\bigg|_{x=0}^{x=l}\right] && (7.10)\\
&= \begin{cases} -\frac{4l}{\pi^2 n^2} & n \text{ is odd}\\ 0 & n \text{ is even}\end{cases} && (7.11)
\end{aligned}$$

so we have the Fourier cosine series of $\phi(x) = x$ given by

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\frac{1}{4}\cos\frac{\pi x}{l} + \frac{1}{9}\cos\frac{3\pi x}{l} + \frac{1}{25}\cos\frac{5\pi x}{l} + \cdots\right).$$

Finally we find the full Fourier series of $\phi(x) = x$ on the interval $[-l, l]$. Recall that

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{l} + B_n\sin\frac{n\pi x}{l}\right).$$

where

$$A_0 = \frac{1}{l}\int_{-l}^{l} x dx = 0$$

$$A_n = \frac{1}{l}\int_{-l}^{l} x\cos\frac{n\pi x}{l}dx = 0, \quad \textit{since it is odd function.}$$

$$B_n = \frac{1}{l}\int_{-l}^{l} x\sin\frac{n\pi x}{l}dx \tag{7.12}$$

$$= \frac{1}{l}\left(-x\frac{l}{m\pi}\cos\frac{m\pi x}{l}\bigg|_{x=-l}^{l} + \int_{l}^{l}\frac{l}{m\pi}\cos\frac{m\pi x}{l}dx\right) \tag{7.13}$$

$$= (-1)^{n+1}\frac{2l}{n\pi}. \tag{7.14}$$

where we see that it is exactly the same as the Fourier sine series on $(0,l)$*. This is because sine and its Fourier sine series are odd.*

Example

Example 22. *We will now use Fourier series to solve the wave equation* $(0<x<l)$ *with boundary conditions :*

$$u_{tt} = c^2u_{xx}, \quad u(0,t) = u(l,t) = 0, \quad u(x,0) = x, u_t(x,0) = 0.$$

Solution 22. *We already know in the previous section (boundary problems), that the solution* $u(x,t)$ *takes the form*

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

hence we have

$$u_t(x,t) = \sum_{n=1}^{\infty}\frac{n\pi c}{l}\left(-A_n\sin\frac{n\pi ct}{l} + B_n\cos\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}.$$

we equate the above with boundary conditions, let $t=0$ *we have*

$$u_t(x,0) = \sum_{n=1}^{\infty}\frac{n\pi c}{l}B_n\sin\frac{n\pi x}{l} = 0$$

so $B_n \equiv 0$*. Also*

$$u(x,0) = \sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{l}$$

and we already know the coefficients of A_n *given from the previous examples. So*

$$u(x,t) = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}.$$

We now derive the complex form of Fourier series. Recall that by *DeMoivre* formula:

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

so one may use $e^{+in\pi x/l}, e^{-in\pi x/l}$ as an alternative pair of sine and cosine.

Definition

Definition 20. *The full Fourier series of $\phi(x)$ defined on $(-l,l)$ is given by*

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

and the coefficients can be computed by

$$c_n = \frac{1}{2l}\int_{-l}^{l} \phi(x) e^{-in\pi x/l} dx$$

Example

Example 23. *A rod has length $l = 1$ and $k = 1$ as a constant. Its temperature satisfies the heat equation $u_t = u_{xx}$. Its left end is held at temperature 0 and right end at 1. Initially the temperature is given by*

$$\phi(x) = \begin{cases} \frac{5}{2}x & 0 < x < \frac{2}{3} \\ 3-2x & \frac{2}{3} < x < 1 \end{cases}.$$

Then what is the solution $u(x,t)$?

Solution 23. *We first find a equilibrium solution, which is the time-independent solution $U(x)$ with the same boundary condition. So in this case $u_t = 0, u_{xx} = 0$, and we easily get that $U(x) = Ax+B$, with boundary condition $U(0) = 0, U(1) = 1$ we have $U(x) = x$ as our equilibrium solution.*

Now consider a function v with $v_t = v_{xx}$, $v(0,t) = v(1,t) = 0$ and $v(x,0) = \phi(x) - U(x)$, then we claim that $u(x,t) = v(x,t) + U(x)$. This is because since v, U both solves the diffusion equation, so does the sum $u = v+U$. The boundary condition is that $u(0,t) = v(0,t) + U(0) = 0, u(1,t) = v(1,t) + U(1) = 1$, and at $t = 0$ we have $u(x,0) = v(x,0) + U(x) = (\phi(x) - U(x)) + U(x) = \phi(x)$ as desired.

Now it suffices to find $v(x,t)$. Recall in the last section we derived that on the interval $(0,1)$, with $k = 1$, we have

$$v(x,t) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \cdot A_n \sin n\pi x$$

where

$$A_n = 2\int_0^1 (\phi(x) - x)\sin(n\pi x) dx, \quad \phi(x) - x = \begin{cases} \frac{3}{2}x & 0 < x < \frac{2}{3} \\ 3-3x & \frac{2}{3} < x < 1 \end{cases}.$$

Hence

$$A_n = 3\int_0^{2/3} x\sin(n\pi x)dx + 6\int_{2/3}^1 \sin(n\pi x)dx - 6\int_{2/3}^1 x\sin(n\pi x)dx.$$

After some simple algebra, we have that

$$A_n = \frac{9}{n^2\pi^2}\sin\frac{2n\pi}{3}.$$

Recall, the sine Fourier series for $\phi(x) = x$ on $(0,1)$ is given by

$$x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\pi n} \sin(n\pi x)$$

hence

$$u(x,t) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \cdot A_n \sin(n\pi x) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\pi n} \sin(n\pi x) \tag{7.15}$$

$$= \sum_{n=1}^{\infty} \left[\frac{9}{\pi^2 n^2} \sin\left(\frac{2n\pi}{3}\right) + (-1)^{n+1} \frac{2}{\pi n} \right] \sin(n\pi) \cdot e^{-n^2\pi^2 t}. \tag{7.16}$$

7.2 Orthogonality and General Fourier Series

Definition

Definition 21. *Let $f, g \in C([a,b])$ be two real-valued functions, their inner product is defined as*

$$\langle f, g \rangle \equiv \int_a^b f(x) g(x) dx$$

and we say $f(x), g(x)$ are orthogonal if $\langle f, g \rangle = 0$.

Definition

Definition 22. *With the inner product defined in the previous definition, we define the L^2 norm of a function $f(x)$ as*

$$||f(x)||^2 := \langle f(x), f(x) \rangle = \int_a^b f^2(x) dx.$$

We wish to study the linear operator

$$A = -\frac{d^2}{dx^2}$$

with some boundary conditions. Some common boundary conditions are as follows:

- **Dirichlet Boundary Condition:** A function $X(x)$ on $[a,b]$ such that $X(a) = X(b) = 0$;
- **Neumann Boundary Condition:** A function $X(x)$ on $[a,b]$ such that $X'(a) = X'(b) = 0$;
- **Periodic Boundary Condition:** A function $X(x)$ on $[a,b]$ such that $X(a) = X(b); X'(a) = X'(b)$.

To generalize the idea of eigenvalues and eigenvectors, we wish to find a constant $\lambda \in \mathbb{R}$ and a non-zero function X satisfying the boundary conditions and such that

$$AX = \lambda X.$$

We say such a function X is an *eigenfunction* and λ is the associated *eigenvalue*. Also, **the eigenfunctions corresponding to different eigenvalues are orthogonal.** To see this, let X_1, X_2 be two different eigenfunctions satisfying the same boundary condition, then

$$-X_1'' = \lambda_1 X_1 \quad -X_2'' = \lambda_2 X_2$$

and we assume $\lambda_1 \neq \lambda_2$, then we need to show

$$\int_a^b X_1 X_2 dx = 0$$

which is equivalent as showing

$$(\lambda_1 - \lambda_2)\int_a^b X_1 X_2 dx = \int_a^b (-X_1'' X_2 + X_2'' X_1) dx = 0$$

the right hand side of the equation above can be simplified using *Green's second identity*, and equipped with boundary conditions, it will simplify to zero in all three cases.

In general, we consider the following boundary conditions:

$$\alpha_1 X(a) + \beta_2 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0$$

$$\alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0$$

Definition

Definition 23. *Such a set of boundary conditions defined as above is called symmetric if*

$$f'(x)g(x) - f(x)g'(x)\Big|_a^b = 0$$

where $f(x), g(x)$ satisfy the boundary condition defined above.

Example

Example 24. *On the interval $(0,l)$, is the boundary condition $u(0,t) = 0, u_x(l,t) = 0$ symmetric?*

Solution 24. *Take f, g be arbitrary test functions such that $f(0) = 0, f'(l) = 0; g(0) = 0, g'(l) = 0$, then*

$$\begin{aligned} f'(x)g(x) - f(x)g'(x)\Big|_a^b &= f'(l)g(l) - f(l)g'(l) - f'(0)g(0) + f(0)g'(0) \\ &\equiv 0. \end{aligned}$$

Hence the boundary condition is symmetric.

Theorem

Theorem 25. *With symmetric boundary condtions, any two eigenfunctions that correspond to different eigenvalues are orthogonal.*

Now let λ_n be the eigenvalue of the eigenfunction $X_n(x)$, we write

$$\phi(x) = \sum_n A_n X_n(x)$$

then

$$\langle \phi(x), X_m(x) \rangle = \sum_n A_n \langle X_n, X_m \rangle = A_m \langle X_m, X_m \rangle$$

by orthogonality.

Definition

Definition 24. *Suppose we have symmetric boundary conditions, and for a function $\phi(x)$ satisfy the form $\phi(x) = \sum_n A_n X_n(x)$, then*

$$A_n = \frac{\langle \phi(x), X_n(x) \rangle}{||X_n(x)||^2} = \frac{\int_a^b \phi(x) X_n(x) dx}{\int_a^b X_n^2 dx}$$

is called the general Fourier series of the function $\phi(x)$.

Theorem

Theorem 26. *All eigenvalues of the eigenvalue problem $-X'' = \lambda X = 0$ over functions $X(x), x \in [a,b]$ satisfying symmetric boundary conditions are real.*

Proof. Assume $\lambda \in \mathbb{C}$ is an eigenvalue of $X(x)$, then by complex conjugate, $\bar{\lambda}$ is also an eigenvalue of $\bar{X}$. Hence

$$\langle \bar{X}, \lambda X \rangle = \langle \bar{\lambda} \bar{X}, X \rangle$$

which means we have

$$\int_a^b (\lambda - \bar{\lambda}) X \bar{X} dx = 0$$

hence $\lambda = \bar{\lambda}$. ■

Theorem

Theorem 27. *Assume for a symmetric boundary condition for $x \in [a,b]$. If $f(x)f'(x)\Big|_a^b \leq 0$ for all functions $f(x)$ satisfying the boundary conditions, then all eigenvalues are positive.*

Proof. Let $X(x)$ be the function satisfying the symmetric boundary conditions as well as the property above, and $-X''(x) = \lambda X(x)$. Then use Green's first identity, we have

$$\int_a^b X''(x) X(x) dx = X'(x) X(x) \Big|_a^b - \int_a^b (X'(x))^2 dx$$

which means

$$-\lambda \int_a^b X^2(x) dx = X'(x) X(x) \Big|_a^b - \int_a^b (X'(x))^2 dx \leq 0$$

hence $\lambda \geq 0$. ■

This also explains why earlier we state $\lambda > 0$ for the Dirichlet boundary condition for wave equation using separation of variables.

Example

Example 25. *Consider the wave equation $u_{tt} = c^2 u_{xx}$ on $0 < x < l$, with initial boundary conidition $u(0,t) = u_x(l,t) = 0$ and $u(x,0) = x$, $u_t(x,0) = 0$. Find the solution explicity in series form.*

Solution 25. *Note that in this wave equation the boundary condition is a bit different! But we may still use separation of variables to find the solutions of X, T given by $u(x,t) = X(x)T(t)$, and hence*

$$u_{tt} = X(x)T''(t) = c^2 u_{xx} = c^2 X''(x)T(t) \implies -\frac{T''(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)} = \lambda$$

we claim that we indeed have a symmetric boundary condition, since

$$\Big(f'(l)g(l) - f(l)g'(l)\Big) - \Big(f'(0)g(0) - f(0)g'(0)\Big) = 0,$$

and by theorem 19,20, all eigenvalues are non-negative. It sufficies to consider positive eigenvalues because otherwise we simply have a trivial solution. We can then solve X, T:

$$T''(t) + \lambda c^2 T(t) = 0 \implies T(t) = A\sin c\sqrt{\lambda}t + B\cos c\sqrt{\lambda}t.$$

$$X''(x) + \lambda X(x) = 0 \implies X(x) = C\sin\sqrt{\lambda}x + D\cos\sqrt{\lambda}x.$$

Now we equate with boundary conditions:

$$u(0,t) = 0 \implies X(0) = 0 \implies D \equiv 0.$$

$$u_x(l,t) = 0 \implies X'(x)\Big|_{x=l} = \sqrt{\lambda}C\cos\sqrt{\lambda}l = 0 \implies \lambda = \frac{(\frac{\pi}{2} + n\pi)^2}{l^2}.$$

So for each n, the eigenvalues and eigenfunctions are

$$\lambda_n = \frac{(\frac{\pi}{2} + n\pi)^2}{l^2}, \quad X_n = \sin\frac{\frac{\pi}{2} + n\pi}{l}x.$$

As for T, we now have

$$T_n(t) = A_n \sin c\frac{\frac{\pi}{2} + n\pi}{l}t + B_n \cos c\frac{\frac{\pi}{2} + n\pi}{l}t,$$

so the solution is

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty}\left(A_n \sin c\frac{\frac{\pi}{2} + n\pi}{l}t + B_n \cos c\frac{\frac{\pi}{2} + n\pi}{l}t\right)\cdot\sin\frac{\frac{\pi}{2} + n\pi}{l}x.$$

We now use boundary conditions to find A_n, B_n, where

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{\frac{\pi}{2} + n\pi}{l}x = x,$$

and the coefficients of B_n can be easily computed by

$$B_n = \frac{\langle x, \sin\frac{\frac{\pi}{2}+n\pi}{l}x\rangle}{\langle \sin\frac{\frac{\pi}{2}+n\pi}{l}x, \sin\frac{\frac{\pi}{2}+n\pi}{l}x\rangle} = \frac{2}{l}\int_0^l x\sin\frac{\frac{\pi}{2}+n\pi}{l}xdx = (-1)^n\cdot\frac{2l}{(n+\frac{1}{2})^2\pi^2}.$$

For another boundary condition $u_t(x,0)=0$, we have

$$\begin{aligned}
u_t(x,0) &= \sum_{n=1}^{\infty}\left(\frac{l}{c(\frac{\pi}{2}+n\pi)}A_n\cos c\frac{\frac{\pi}{2}+n\pi}{l}\times 0 - \frac{l}{c(\frac{\pi}{2}+n\pi)}B_n\sin c\frac{\frac{\pi}{2}+n\pi}{l}\times 0\right)\cdot\sin\frac{\frac{\pi}{2}+n\pi}{l}x \\
&= \sum_{n=1}^{\infty}\frac{l}{c(\frac{\pi}{2}+n\pi)}A_n\sin\frac{\frac{\pi}{2}+n\pi}{l}x,
\end{aligned}$$

hence $A_n\equiv 0$. So the general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty}(-1)^n\cdot\frac{2l}{(n+\frac{1}{2})^2\pi^2}\cos c\left(\frac{\frac{\pi}{2}+n\pi}{l}t\right)\cdot\sin\left(\frac{\frac{\pi}{2}+n\pi}{l}x\right).$$

Theorem

Theorem 28 (Gram-Schmidt Orthogonalization)**.** *Let $\{X_n\}$ be a sequence of mutually linearly independent vectors in an inner product space $(V,\langle\cdot,\cdot\rangle)$, then we may get a sequence $\{Z_n\}$ of mutually orthonormal vectors by the following procedure:*

- ***Step 1:*** *Define*

$$Z_1 := \frac{X_1}{||X_1||};$$

- ***Step 2:*** *For all $X_i, i>1$, define*

$$Y_i = X_i - \sum_{j=1}^{i-1}\langle X_i, Z_j\rangle Z_j;$$

where

$$Z_j = \frac{Y_i}{||Y_i||},$$

and we normalize Y_i.

This theorem is especially useful in linear algebra for constructing ortonormal basis for an inner product space.

Proof. First, it is easy to see that $||Z_i||=1$. By induction, when $n=1$ the result is trivial. For $n>1$, any $k<n$, we have

$$\begin{aligned}
\langle Z_n, Y_k\rangle &= \left\langle Z_n, X_k - \sum_{j=1}^{k-1}\langle X_k, Z_j\rangle Z_j\right\rangle \\
&= \langle Z_n, X_k\rangle - \sum_{j=1}^{k-1}\langle X_k, Z_j\rangle\langle Z_n, Z_j\rangle \\
&= \langle Z_n, X_k\rangle - \langle X_n, Z_k\rangle \quad \text{(since } \{Z_1,\cdots,Z_{n-1}\} \text{ are orthonormal)} \\
&= 0.
\end{aligned}$$

■

Theorem

Theorem 29 (Cauchy-Schwarz Inequality)**.** *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, $\mathbf{u}, \mathbf{v} \in V$, then*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||.$$

Proof. Consider a function $g(t) = ||\mathbf{u} + t\mathbf{v}||^2$ where $g(t) \geq 0$, then

$$g(t) = ||\mathbf{v}||^2 t^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle t + ||\mathbf{u}||^2.$$

Note that $g(t)$ is a quadraic equation so the minimizer is obtained at

$$t_{\min} = -\frac{b}{2a} = -\frac{2\langle \mathbf{u}, \mathbf{v} \rangle}{2||\mathbf{v}||^2},$$

and by definition $g(t_{\min}) \geq 0$ so we have

$$||\mathbf{v}||^2 \cdot \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{||\mathbf{v}||^4} - \frac{2|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{||\mathbf{v}||^2} + ||\mathbf{u}||^2 \geq 0,$$

which is Cauchy-Schwarz inequality:

$$||\mathbf{u}||^2 \cdot ||\mathbf{v}||^2 \geq |\langle \mathbf{u}, \mathbf{v} \rangle|^2.$$

■

7.3 Fourier Series Corresponding to Different Boundary Conditions

7.3.1 Dirichlet Boundary Condition:

Let us first consider the eigenvalue problem with Dirichlet boundary condition on $[0, l]$ given by

$$X'' + \lambda X = 0, \quad X(0) = X(l) = 0$$

since we know that any symmetric boundary condition has real eigenvalues, hence we consider three conditions:

- If $\lambda < 0$: In this case we have a general solution to the ODE given by

  $$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}, \quad A, B \in \mathbb{R}.$$

  By imposing the boundary condition, $X(0) = X(l) = 0$ we have

  $$A + B = 0 \quad Ae^{\sqrt{-\lambda}l} + Be^{-\sqrt{-\lambda}l} = 0,$$

  which means

  $$B = -A, \quad A\left(e^{\sqrt{-\lambda}l} - e^{-\sqrt{-\lambda}l}\right) = 0,$$

  hence $A = B = 0$, which means there are no non-trivial solutions.

- If $\lambda = 0$, in this case the general solution takes the form $X(x) = Ax + B, A, B \in \mathbb{R}$ and the boundary condition will yield $A = B = 0$, which is again we do not have non-trivial solution.
- If $\lambda < 0$, then the general solution is

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x, \quad A, B \in \mathbb{R}.$$

with boundary condition imposed, we have $A = 0$. Also $B = 0$ or $\sin\sqrt{\lambda}l = 0$. We take the latter non-trivial solution, and it holds for $\lambda = n^2\pi^2/l^2$, and the corresponding eigenvalues are $\sin n\pi x/l$. So we have the following:

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \quad X_n(x) = \sin\frac{n\pi x}{l}.$$

Hence the Fourier series for Dirichlet boundary condition is given by

$$A_n = \frac{\langle \phi, X_n \rangle}{||X_n||^2} = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

The corresponding Fourier series is

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}.$$

7.3.2 Neumann Boundary Condition:

The Neumann Boundary Condition on $[0, l]$ takes the form

$$X'(0) = X'(l) = 0.$$

It can be computed that the eigenvalues and eigenfunctions are

$$X_n(x) = \cos\frac{n\pi x}{l} \quad \lambda_n = \frac{n^2\pi^2}{l^2}.$$

Hence the coefficients of the Fourier series are given by

$$A_n = \frac{\langle \phi, X_n \rangle}{||X_n||^2} = \frac{2}{l}\int_0^l \phi(x)\cos\frac{n\pi x}{l}dx$$

with the Fourier series given by

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}.$$

7.3.3 Periodic Boundary Condition:

The periodic boundary condition on $[-l, l]$ is given by

$$X(l) = X(l), \quad X'(l) = X'(-l).$$

With this condition, we have two eigenfunctions:

$$Y_n(x) = \cos\frac{n\pi x}{l}, \quad , Z_n(x) = \sin\frac{n\pi x}{l},$$

corresponding to eigenvalues $\lambda_n = \frac{n^2\pi^2}{l^2}$. Hence the Fourier series is

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right),$$

where

$$A_n = \frac{1}{l}\int_{-l}^{l}\phi(x)\cos\frac{n\pi x}{l}dx, \quad B_n = \frac{1}{l}\int_{-l}^{l}\phi(x)\sin\frac{n\pi x}{l}dx.$$

7.4 Convergence of Fourier Series

We first introduce different modes of convergence:

Definition

Definition 25. *Consider a sequence of functions $\{f_n\}_{n=1}^{\infty}$ defined on $[a,b]$.*

- *f_n converges to f pointwise, if and only if for every $x \in [a,b]$, $f_n(x) \to f(x)$ as $n \to \infty$.*
- *f_n converges to f uniformly, if and only if*

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$

- *f_n converges to f in L^2 if and only if*

$$\int_a^b |f_n(x) - f(x)|^2 dx \overset{n\to\infty}{\to} 0.$$

In the sence of Fourier series $S_n = \sum_n A_n X_n$, the pointwise convergence will always hold. How can we justify the other types of convergence? The following theorem tells.

Theorem

Theorem 30. *Let $f(x)$ be defined on $[a,b]$, and $S_n = \sum A_n X_n$ be the Fourier series of $f(x)$, then:*

- *S_n converges to $f(x)$ uniformly, provided that $f(x), f'(x), f''(x)$ exist, are continuous on $[a,b]$ and $f(x)$ satisfies the symmetric boundary conditions.*
- *S_n converges to $f(x)$ in L^2, provided that*

$$\int_a^b |f(x)|^2 dx < \infty.$$

Next, we present convergence for full Fourier series. We say a function $f(x)$ is piecewise continuous on an interval $[a,b]$ if it is continuous at all but a finite number of points and has jump dicontinuities at all those points.

Theorem

Theorem 31. *The classical full Fourier series convergences to $f(x)$ pointwise on $[a,b]$ provided that $f(x)$ is continuous on $[a,b]$ and $f'(x)$ is piecewise continuous on $[a,b]$. More generally if $f(x)$ is onlypiecewise continuous on $[a,b]$ and $f'(x)$ is also piecewise continuous on $[a,b]$, then the classical Fourier series converges at every point x where*

$$\sum_n A_n X_n(x) = \frac{1}{2}[f(x^+) + f(x^-)].$$

We may view this theorem as follows: At a jump discontinuity the series converges to the average of the limits from the right and from the left.

Theorem

Theorem 32 (Bessel's Inequality)**.** *Let $\phi(x) \in L^2((a,b))$ and suppose $\{X_n(x)\}$ is an orthogonal family of functions on (a,b). Define*

$$A_n := \frac{\langle \phi(x), X_n(x) \rangle}{||X_n(x)||^2},$$

then

$$\sum_{n=1}^{\infty} A_n^2 \int_a^b |X_n(x)|^2 dx \leq \int_a^b |\phi(x)|^2 dx.$$

Proof. Let $\{c_n\}$ be a sequence of real numbers, define

$$\begin{aligned}
E_N &:= \int_a^b \left| \phi(x) - \sum_{n=1}^{N} c_n X_n(x) \right|^2 dx \\
&= \int_a^b |\phi(x)|^2 dx - 2\sum_{n=1}^{N} c_n \int_a^b \phi(x) X_n(x) dx + \sum_{m=1}^{N}\sum_{n=1}^{N} c_m c_n \int_a^b X_m(x) X_n(x) dx \\
&= ||\phi(x)||^2 - 2\sum_{n=1}^{N} c_n \langle \phi(x), X_n(x) \rangle + \sum_{n=1}^{N} c_n^2 ||X_n||^2 \quad \text{by orthogonality of } \{X_n\}
\end{aligned}$$

View above as a function of each c_n, we see that it is then a quadratic function: $f(c_n) = a_n c_n^2 + b_n c_n + d_n$ and each c_n is minimized at

$$c_n = -\frac{b_n}{2a_n} = \frac{\langle \phi(x), X_n(x) \rangle}{||X_n(x)||^2}.$$

With c_n minimized as above, we have

$$E_N = ||\phi(x)||^2 - \sum_{n=1}^{N} A_n^2 ||X_n(x)||^2.$$

By definition, $E_N \geq 0$, so

$$||\phi(x)||^2 \geq \sum_{n=1}^{N} A_n^2 ||X_n(x)||^2$$

which is,

$$\sum_{n=1}^{N} A_n^2 \int_a^b |X_n(x)|^2 dx \leq \int_a^b |\phi(x)|^2 dx$$

and the Bessel's inequaity is proven by taking $N \to \infty$. ■

Theorem

Theorem 33 (Parseval's Equality)**.** *Let $\phi(x) \in L^2((a,b))$, and $X_n(x), A_n(x)$ be the eigenfunctions and Fourier coefficients of any general Fourier series for $\phi(x)$ respectively. Then*

$$\sum_{n=1}^{\infty} A_n^2 \int_a^b |X_n(x)|^2 dx = \int_a^b |\phi(x)|^2 dx.$$

We will not prove this theorem.

Let $\phi(x) \equiv 1 \in L^2((0,\pi))$, using classical Fourier sine series defined on $(0,\pi)$, we have

$$X_n(x) = \sin\frac{n\pi x}{l} = \sin(nx)$$

and

$$A_n = \frac{\langle \phi(x), X_n(x) \rangle}{||X_n(x)||^2} = \frac{\int_0^\pi \sin(nx)dx}{\int_0^\pi \sin^2(nx)dx} = \frac{4}{n\pi} \quad \text{for odd } n.$$

hence using Parseval's equality, we have

$$\sum_{\text{odd } n} \left(\frac{4}{n\pi}\right)^2 \cdot \int_0^\pi \sin^2(nx)dx = \int_0^\pi 1^2 dx$$

which is

$$\sum_{\text{odd } n} \left(\frac{4}{n\pi}\right)^2 \cdot \frac{\pi}{2} = \pi$$

i.e,

$$\sum_{\text{odd } n} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Theorem

Theorem 34 (Riemann-Lebesgue Lemma)**.** *If $\phi(x) \in L^2((a,b))$, then the classical Fourier coefficients of $\phi(x)$ goes to 0 as $n \to \infty$.*

By Riemann-Lebesgue lemma, recall the Dirichlet boundary conditions on $[0,l]$, we have that every $\phi(x)$ defined on $(0,l)$ with $||\phi(x)||^2 < \infty$ satisfies

$$\int_0^l \phi(x) \sin\frac{n\pi x}{l} dx \to 0, \quad \text{as } n \to \infty.$$

Since C_C^∞ functions are surely square integrabe, so the above equation also means

$$\sin\frac{n\pi x}{l} \to 0 \quad \text{in distribution.}$$

Theorem

Theorem 35. *Suppose $\phi \in L^2((a,b))$, then its general Fourier series associated with any symmetric boundary conditions converges to ϕ in L^2 on (a,b). We say the eigenfunctions assciated with a symmetric boundary condition are complete in $L^2((a,b))$.*

We shall omit the proof for this theorem.

Definition

Definition 26. *A sequence of functions $\{f_n\}$ defined on $[a,b]$ converges to f pointwise, if and only if for every $x \in [a,b]$, $f_n(x) \to f(x)$ as $n \to \infty$.*

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