Linear Algebra Sample Examination

PART (i)

Multiple Choice Questions : Each question will have exactly one correct answer.

1. In $\mathbb{R}^3$, a plane $\mathcal{W}$ is given by $\mathcal{W} := \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right)$, then what is $\mathcal{W}^{\perp}$?

(A) $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$ (B) $\text{span}\left(\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}\right)$ (C) $\text{span}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right)$ (D) $\text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)$

2. Let $\mathcal{V}$ be a vector space over the field $\mathbb{R}$, $\mathcal{T} : \mathcal{V} \longrightarrow \mathcal{V}$ is a linear transformation. Given that $\ker(\mathcal{T}) = \{\mathbf{0}\}$, then which of the following **is not** necessarily true?

(A) $\mathcal{V} = \ker(\mathcal{T}) \oplus \text{Im}(\mathcal{T})$ (B) $\mathcal{T}$ is diagonalizable

(C) **Rank** $(\mathcal{T}) = \dim(\mathcal{V})$ (D) $\mathcal{T}$ is bijective

3. Let $a \in \mathbb{R}$, in the following 3×3 matrices, which one **is not** diagonalizable?

(A) $\begin{bmatrix} 1 & a & -1 \\ 0 & 2 & a \\ 0 & 0 & 3 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & a & 0 \\ a & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 1 & a \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 1 & a \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

4. Let $\mathcal{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation, and let $\mathcal{E} := \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be the standard ordered basis. Given that $\mathcal{T}(\boldsymbol{e}_1) = \boldsymbol{e}_2 + \boldsymbol{e}_3; \mathcal{T}(\boldsymbol{e}_2) = \boldsymbol{e}_1 + \boldsymbol{e}_3; \mathcal{T}(\boldsymbol{e}_3) = \boldsymbol{e}_1 + \boldsymbol{e}_2$. Let $\boldsymbol{v} \in \mathbb{R}^3$, what is the value of $\mathcal{T}^5(\boldsymbol{v})$ ($\boldsymbol{v}$ is represented under the basis $\mathcal{E}$)?

(A) $5\ \boldsymbol{v}$ (B) $10\ \boldsymbol{v}$ (C) $16\ \boldsymbol{v}$ (D) $32\ \boldsymbol{v}$

5. Let $\mathcal{W}$ be the vector space of all polynomials with maximum degree 2 with real coefficients, then which one **is not** a subspace of $\mathcal{W}$?

(A) $\mathcal{W}_1 := \{f(x) \in \mathbb{P}_2(\mathbb{R}) : f(1) = 0\}$

(B) $\mathcal{W}_2 := \{f(x) \in \mathbb{P}_2(\mathbb{R}) : f(x) + f(-x) = 0\}$

(C) $\mathcal{W}_3 := \{f(x) \in \mathbb{P}_2(\mathbb{R}) : f'(x) = x\}$

(D) $\mathcal{W}_4 := \{f(x) \in \mathbb{P}_2(\mathbb{R}) : \int_{-1}^{1} f(x)dx = 0\}$

6. Let $\mathcal{T} : \mathcal{V} \longrightarrow \mathcal{V}$ be a linear transformation, where $\mathcal{V}$ is a vector space over the field $\mathbb{R}$ and $\dim(\mathcal{V}) = 5$. Then which statement **does not conclude** that $\mathcal{T}$ is diagonalizable?

(A) $\mathcal{T} \circ \mathcal{T} = \mathcal{T}$

(B) $\mathcal{T}$ has 5 distinct eigenvalues

(C) $\mathcal{T}$ has 2 distinct eigenvalues and $\mathcal{T} := \mathcal{W} \oplus \mathcal{W}^{\perp}$ for a subspace $\mathcal{W} \subset \mathcal{V}$

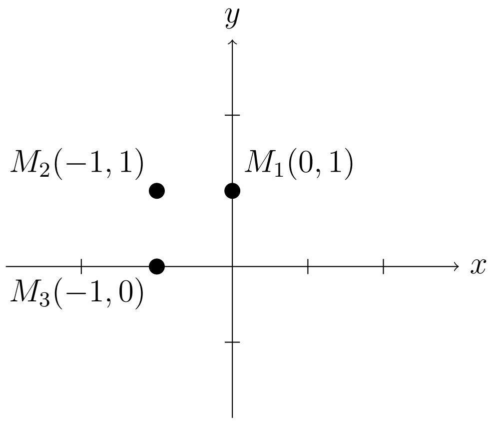
(D) $\mathcal{T}$ has 3 distinct eigenvalues including 0, and $\dim(\ker(\mathcal{T})) = 3$

7. Let $\mathcal{V}$ be an inner product space over the field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$). Let $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$ and $\alpha \in \mathbb{F}$. Then what is the correct simplification of $\|\alpha \boldsymbol{u} + i\boldsymbol{v}\|^2$?

(A) $\alpha^2\|\boldsymbol{u}\|^2 + \overline{\alpha}i\langle \boldsymbol{v}, \boldsymbol{u}\rangle - \alpha i\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{v}\|^2$ (B) $\alpha^2\|\boldsymbol{u}\|^2 + \alpha i\langle \boldsymbol{v}, \boldsymbol{u}\rangle - \alpha i\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{v}\|^2$

(C) $\alpha^2\|\boldsymbol{u}\|^2 + \alpha i\langle \boldsymbol{v}, \boldsymbol{u}\rangle - \alpha i\langle \boldsymbol{u}, \boldsymbol{v}\rangle - \|\boldsymbol{v}\|^2$ (D) $\alpha^2\|\boldsymbol{u}\|^2 + \overline{\alpha}i\langle \boldsymbol{v}, \boldsymbol{u}\rangle - \alpha i\langle \boldsymbol{u}, \boldsymbol{v}\rangle - \|\boldsymbol{v}\|^2$

8. In $\mathbb{R}^2$, given the datapoints $M_1 = (0, 1); M_2 = (-1, 1); M_3 = (-1, 0)$, and according to model matching the equation for this non-linear regression is of the form $ax^2 + by^2 - (x + y) - 1 = 0$. Then the equation of the least square regression should be



(A) $\frac{1}{3}x^2+\frac{5}{3}y^2-(x+y)-1=0$

(B) $-\frac{1}{3}x^2+\frac{5}{3}y^2-(x+y)-1=0$

(C) $\frac{1}{3}x^2-\frac{5}{3}y^2-(x+y)-1=0$

(D) $-\frac{1}{3}x^2-\frac{5}{3}y^2-(x+y)-1=0$

9. Let $\mathbf{X_1}=\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}$; $\mathbf{X_2}=\begin{bmatrix}0 & \frac{1}{2}\\ 1 & 0\end{bmatrix}$; $\mathbf{X_3}=\begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix}$, if there exists a 2×2 matrix $\mathbf{A}$ such that $\mathbf{AX_1X_2X_3}=\begin{bmatrix}2 & -2\\ 4 & 3\end{bmatrix}$, then what is $\mathbf{A}$?

(A) $\begin{bmatrix}2 & \frac{3}{2}\\ 4 & -\frac{1}{2}\end{bmatrix}$

(B) $\begin{bmatrix}-3 & -1\\ 1 & -2\end{bmatrix}$

(C) $\begin{bmatrix}-6 & -\frac{1}{2}\\ 2 & -1\end{bmatrix}$

(D) $\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}$

10. Suppose $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ are linearly independent, then which of the following **is not** linearly independent?

(A) $\{\boldsymbol{u}-\boldsymbol{v},\boldsymbol{v}-\boldsymbol{w},\boldsymbol{w}-\boldsymbol{u}\}$

(B) $\{\boldsymbol{u}+\boldsymbol{v},\boldsymbol{v}+\boldsymbol{w},\boldsymbol{w}+\boldsymbol{u}\}$

(C) $\{\boldsymbol{u}-\boldsymbol{v},\boldsymbol{v}-\boldsymbol{w},\boldsymbol{u}-\boldsymbol{w}\}$

(D) $\{\boldsymbol{u}+\boldsymbol{v},\boldsymbol{v}-\boldsymbol{w},\boldsymbol{w}-\boldsymbol{u}\}$

PART (ii)

Long Answer Questions : Each question requires detailed calculations and proofs

11. Calculate the determinant of the matrix given by

$$\mathcal{Q} = \begin{bmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{bmatrix}$$

where $x \in \mathbb{R}$, simplify as much as possible.

12. In $\mathbb{R}^3$, a subspace $\mathcal{W}$ is given by the plane $x + 2y - z = 0$.

(12 - i) By applying *Gram-Schmidt* algorithm, construct an **orthonormal** basis for $\mathcal{W}^{\perp}$.

(12 - ii) Extend the basis of $\mathcal{W}^{\perp}$ by one vector, such that the new basis form an **orthonormal** basis for $\mathbb{R}^3$, denote this basis to be β.

13. Suppose a plane in $\mathbb{R}^3$ to be $x + y + z = 0$. Let $\mathcal{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ to be a reflection in this plane (mirror image). Find a formula for $\mathcal{T}(\boldsymbol{v})$ under the standard basis of $\mathbb{R}^3$ given by $\mathcal{E} := \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$.

14. Let $\mathcal{V}$ be a finite dimensional vector space over a field $\mathbb{F}$, a linear transformation $\mathcal{T} : \mathbb{P}_n(\mathbb{R}) \longrightarrow \mathbb{P}_n(\mathbb{R})$ is given by $\mathcal{T}(p(x)) = (1 + x)p'(x)$. Let $\alpha := \{1, x, x^2, \cdots, x^n\}$ to be a basis for $\mathbb{P}_n(\mathbb{R})$. Prove that $\mathcal{T}$ is diagonalizable.

15. Let $\mathcal{V}$ be a finite dimensional inner product space over a field $\mathbb{F}$, and $\mathcal{W} \subset \mathcal{V}$. A linear transformation $\mathcal{T} : \mathcal{V} \longrightarrow \mathcal{V}$ is the orthogonal projection of $\mathcal{V}$ on $\mathcal{W}$. Prove that $\mathcal{I}_{\mathcal{V}} - \mathcal{T}$ is the orthogonal projection of $\mathcal{V}$ on $\mathcal{W}^{\perp}$. ($\mathcal{I}_{\mathcal{V}}$ represents the identity transformation)

Above are challenging questions and they require way more work! If you are passionate about linear algebra sit down with a coffee and get ready to spend some time on them!

16. Let $\mathcal{V}$ be an inner product space with dimension n, and $\beta := \{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n\}$ is an orthonormal basis of $\mathcal{V}$. Let $\mathcal{T} : \mathcal{V} \longrightarrow \mathcal{V}$ be a linear transformation, then prove that we have

$$[\mathcal{T}]_\beta = \begin{bmatrix} \langle \mathcal{T}\boldsymbol{e}_1, \boldsymbol{e}_1 \rangle & \cdots & \cdots & \langle \mathcal{T}\boldsymbol{e}_n, \boldsymbol{e}_1 \rangle \\ \langle \mathcal{T}\boldsymbol{e}_1, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \mathcal{T}\boldsymbol{e}_n, \boldsymbol{e}_2 \rangle \\ \vdots & & & \vdots \\ \langle \mathcal{T}\boldsymbol{e}_1, \boldsymbol{e}_n \rangle & \cdots & \cdots & \langle \mathcal{T}\boldsymbol{e}_n, \boldsymbol{e}_n \rangle \end{bmatrix}$$

For each $\langle \mathcal{T}\boldsymbol{e}_i, \boldsymbol{e}_j \rangle$, it is called the *Fourier Coefficients.*

17. Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$ where $\dim(\mathcal{V}) = n$, a linear transformation $\mathcal{T} : \mathcal{V} \longrightarrow \mathcal{V}$ has $\boldsymbol{v}$ as an eigenvalue. If the $\mathcal{T}$-cyclic subspace of $\mathcal{V}$ generated by $\{\boldsymbol{v}, \mathcal{T}(\boldsymbol{v}), \mathcal{T}^2(\boldsymbol{v}), \cdots, \mathcal{T}^{n-1}(\boldsymbol{v})\}$ is a basis for $\mathcal{V}$, then we know that there is a unique linear combination such that

$$\mathcal{T}^n(\boldsymbol{v}) + c_0\boldsymbol{v} + c_1\mathcal{T}(\boldsymbol{v}) + \cdots + c_{n-1}\mathcal{T}^{n-1}(\boldsymbol{v}) = 0$$

where $c_i \in \mathbb{F}$. Prove that each c_i $(0 \leq i \leq n-1)$ is exactly the coefficients of x^i in the characteristic polynomial of $\mathcal{T}$. The proof **should not** use *Cayley-Hamilton Theorem*, in fact we need to prove this theorem.