

Linear Algebra Sample Examination

PART (i)

Multiple Choice Questions : Each question will have exactly one correct answer.

1. In $\mathbb{R}^3$, a plane $\mathcal{W}$ is given by $\mathcal{W} := \text{span} \left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right)$, then what is $\mathcal{W}^\perp$?

(A) $\text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$

(B) $\text{span} \left(\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right)$

(C) $\text{span} \left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right)$

(D) $\text{span} \left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)$

2. Let $\mathcal{V}$ be a vector space over the field $\mathbb{R}$, $\mathcal{T} : \mathcal{V} \rightarrow \mathcal{V}$ is a linear transformation. Given that $\ker(\mathcal{T}) = \{\mathbf{0}\}$, then which of the following **is not** necessarily true?

(A) $\mathcal{V} = \ker(\mathcal{T}) \oplus \text{Im}(\mathcal{T})$

(B) $\mathcal{T}$ is diagonalizable

(C) **Rank** $(\mathcal{T}) = \dim(\mathcal{V})$

(D) $\mathcal{T}$ is bijective

3. Let $a \in \mathbb{R}$, in the following 3×3 matrices, which one **is not** diagonalizable?

(A) $\begin{bmatrix} 1 & a & -1 \\ 0 & 2 & a \\ 0 & 0 & 3 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & a & 0 \\ a & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 1 & a \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 1 & a \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

4. Let $\mathcal{T} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation, and let $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard ordered basis. Given that $\mathcal{T}(\mathbf{e}_1) = \mathbf{e}_2 + \mathbf{e}_3$; $\mathcal{T}(\mathbf{e}_2) = \mathbf{e}_1 + \mathbf{e}_3$; $\mathcal{T}(\mathbf{e}_3) = \mathbf{e}_1 + \mathbf{e}_2$. Let $\mathbf{v} \in \mathbb{R}^3$, what is the value of $\mathcal{T}^5(\mathbf{v})$ ($\mathbf{v}$ is represented under the basis $\mathcal{E}$)?

(A) $5 \mathbf{v}$

(B) $10 \mathbf{v}$

(C) $16 \mathbf{v}$

(D) $32 \mathbf{v}$

5. Let $\mathcal{W}$ be the vector space of all polynomials with maximum degree 2 with real coefficients,

then which one **is not** a subspace of $\mathcal{W}$?

(A) $\mathcal{W}_1 := \{f(x) \in \mathbb{P}_2(\mathbb{R}) : f(1) = 0\}$

(B) $\mathcal{W}_2 := \{f(x) \in \mathbb{P}_2(\mathbb{R}) : f(x) + f(-x) = 0\}$

(C) $\mathcal{W}_3 := \{f(x) \in \mathbb{P}_2(\mathbb{R}) : f'(x) = x\}$

(D) $\mathcal{W}_4 := \{f(x) \in \mathbb{P}_2(\mathbb{R}) : \int_{-1}^1 f(x)dx = 0\}$

6. Let $\mathcal{T} : \mathcal{V} \rightarrow \mathcal{V}$ be a linear transformation, where $\mathcal{V}$ is a vector space over the field $\mathbb{R}$ and $\dim(\mathcal{V}) = 5$. Then which statement **does not conclude** that $\mathcal{T}$ is diagonalizable?

(A) $\mathcal{T} \circ \mathcal{T} = \mathcal{T}$

(B) $\mathcal{T}$ has 5 distinct eigenvalues

(C) $\mathcal{T}$ has 2 distinct eigenvalues and $\mathcal{T} := \mathcal{W} \oplus \mathcal{W}^\perp$ for a subspace $\mathcal{W} \subset \mathcal{V}$

(D) $\mathcal{T}$ has 3 distinct eigenvalues including 0, and $\dim(\ker(\mathcal{T})) = 3$.

7. Let $\mathcal{V}$ be an inner product space over the field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$). Let $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ and $\alpha \in \mathbb{F}$. Then what is the correct simplification of $\|\alpha\mathbf{u} + i\mathbf{v}\|^2$?

(A) $\alpha^2\|\mathbf{u}\|^2 + \bar{\alpha}i\langle\mathbf{v}, \mathbf{u}\rangle - \alpha i\langle\mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2$ (B) $\alpha^2\|\mathbf{u}\|^2 + \alpha i\langle\mathbf{v}, \mathbf{u}\rangle - \alpha i\langle\mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2$

(C) $\alpha^2\|\mathbf{u}\|^2 + \alpha i\langle\mathbf{v}, \mathbf{u}\rangle - \alpha i\langle\mathbf{u}, \mathbf{v}\rangle - \|\mathbf{v}\|^2$ (D) $\alpha^2\|\mathbf{u}\|^2 + \bar{\alpha}i\langle\mathbf{v}, \mathbf{u}\rangle - \alpha i\langle\mathbf{u}, \mathbf{v}\rangle - \|\mathbf{v}\|^2$

8. In $\mathbb{R}^2$, given the datapoints $M_1 = (0, 1); M_2 = (-1, 1); M_3 = (-1, 0)$, and according to model matching the equation for this non-linear regression is of the form $ax^2 + by^2 - (x + y) - 1 = 0$. Then the equation of the least square regression should be

(A) $\frac{1}{3}x^2 + \frac{5}{3}y^2 - (x + y) - 1 = 0$

(B) $-\frac{1}{3}x^2 + \frac{5}{3}y^2 - (x + y) - 1 = 0$

(C) $\frac{1}{3}x^2 - \frac{5}{3}y^2 - (x + y) - 1 = 0$

(D) $-\frac{1}{3}x^2 - \frac{5}{3}y^2 - (x + y) - 1 = 0$

9. Let $\mathbf{X}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$; $\mathbf{X}_2 = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$; $\mathbf{X}_3 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, if there exists a 2×2 matrix $\mathbf{A}$ such that

$\mathbf{A}\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3 = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$, then what is $\mathbf{A}$?

(A) $\begin{bmatrix} 2 & \frac{3}{2} \\ 4 & -\frac{1}{2} \end{bmatrix}$

(B) $\begin{bmatrix} -3 & -1 \\ 1 & -2 \end{bmatrix}$

(C) $\begin{bmatrix} -6 & -\frac{1}{2} \\ 2 & -1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

10. Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, then which of the following is **not** linearly independent?

(A) $\{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{w} - \mathbf{u}\}$

(B) $\{\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}, \mathbf{w} + \mathbf{u}\}$

(C) $\{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{u} - \mathbf{w}\}$

(D) $\{\mathbf{u} + \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{w} - \mathbf{u}\}$

PART (ii)

Long Answer Questions : Each question requires detailed calculations and proofs

11. Calculate the determinant of the matrix given by

$$\mathcal{Q} = \begin{bmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{bmatrix}$$

where $x \in \mathbb{R}$, simplify as much as possible.

12. In $\mathbb{R}^3$, a subspace $\mathcal{W}$ is given by the plane $x + 2y - z = 0$.

(12 - i) By applying *Gram-Schmidt* algorithm, construct an **orthonormal** basis for $\mathcal{W}^\perp$.

(12 - ii) Extend the basis of $\mathcal{W}^\perp$ by one vector, such that the new basis form an **orthonormal** basis for $\mathbb{R}^3$, denote this basis to be β .

13. Suppose a plane in $\mathbb{R}^3$ to be $x + y + z = 0$. Let $\mathcal{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ to be a reflection in this plane (mirror image). Find a formula for $\mathcal{T}(\mathbf{v})$ under the standard basis of $\mathbb{R}^3$ given by $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

14. Let $\mathcal{V}$ be a finite dimensional vector space over a field $\mathbb{F}$, a linear transformation $\mathcal{T} : \mathbb{P}_n(\mathbb{R}) \longrightarrow \mathbb{P}_n(\mathbb{R})$ is given by $\mathcal{T}(p(x)) = (1 + x)p'(x)$. Let $\alpha := \{1, x, x^2, \dots, x^n\}$ to be a basis for $\mathbb{P}_n(\mathbb{R})$. Prove that $\mathcal{T}$ is diagonalizable.

15. Let $\mathcal{V}$ be a finite dimensional inner product space over a field $\mathbb{F}$, and $\mathcal{W} \subset \mathcal{V}$. A linear transformation $\mathcal{T} : \mathcal{V} \longrightarrow \mathcal{V}$ is the orthogonal projection of $\mathcal{V}$ on $\mathcal{W}$. Prove that $\mathcal{I}_{\mathcal{V}} - \mathcal{T}$ is the orthogonal projection of $\mathcal{V}$ on $\mathcal{W}^\perp$. ($\mathcal{I}_{\mathcal{V}}$ represents the identity transformation)

16. Let $\mathcal{V}$ be an inner product space with dimension n , and $\beta := \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is an orthonormal basis of $\mathcal{V}$. Let $\mathcal{T} : \mathcal{V} \longrightarrow \mathcal{V}$ be a linear transformation, then prove that we have

$$[\mathcal{T}]_\beta = \begin{bmatrix} \langle \mathcal{T}\mathbf{e}_1, \mathbf{e}_1 \rangle & \cdots & \cdots & \langle \mathcal{T}\mathbf{e}_n, \mathbf{e}_1 \rangle \\ \langle \mathcal{T}\mathbf{e}_1, \mathbf{e}_2 \rangle & \cdots & \cdots & \langle \mathcal{T}\mathbf{e}_n, \mathbf{e}_2 \rangle \\ \vdots & & & \vdots \\ \langle \mathcal{T}\mathbf{e}_1, \mathbf{e}_n \rangle & \cdots & \cdots & \langle \mathcal{T}\mathbf{e}_n, \mathbf{e}_n \rangle \end{bmatrix}$$

For each $\langle \mathcal{T}\mathbf{e}_i, \mathbf{e}_j \rangle$, it is called the *Fourier Coefficients*.